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1. **(15pt)** Consider the function
$$f(x, y) = x^2 - y^3 + 4.$$

(a) Find an equation of the plane tangent to the graph of the function at the point $(1, 2, -3)$.

(b) Use the linear approximation to the function to estimate $f(0.95, 2.05)$. Your answer should be a single number in decimal form, or written as a reduced fraction.

a) $\nabla f = \langle 2x, -3y^2 \rangle$

$\nabla f(1,2) = \langle 2\cdot 1, -3\cdot 4 \rangle = \langle 2, -12 \rangle$

$z = 2(x-1) - 12(y-2) - 3$

b) $L(x,y) = 2(x-1) - 12(y-2) - 3$

$L(0.95, 2.05) = 2\cdot(0.95-1) - 12(2.05-2) - 3 =$

$-2\cdot 0.05 - 12\cdot 0.05 - 3 = 0.10 - 0.6 - 3 =$

$-3.7$

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2. **(15 pt)** Find all critical points of the function

$$f(x, y) = x^2y - 2xy - 5x^2 + 10x$$

and classify them using the Second Derivative Test.

$f_x = 2xy - 2y - 10x + 10 = 0$

$f_y = x^2 - 2x = 0$

$x^2 - 2x = x(x-2) = 0, \quad x = 0 \quad \text{or} \quad x = 2$

$x = 0$: $\quad -2y + 10 = 0$, $y = 5$, critical point $(0,5)$

$x = 2$: $\quad 4y - 2y - 20 + 10 = 0$, $2y - 10 = 0$, $y = 5$, critical point $(2,5)$

$f_{xx} = 2y - 10 \qquad f_{xy} = 2x - 2 \qquad f_{yy} = 0$

$D(x,y) = (2y-10)\cdot 0 - (2x-2)^2 = -(2x-2)^2$

$f(0,5) = -4 < 0 \qquad (0,5)$ is a saddle point

$f(2,5) = -(2\cdot 2-2)^2 < 0$, $(2,5)$ is a saddle point

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3. **(15 pt)** Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x, y) = x + y$ subject to the constraint

$$x^2 + 4y^2 = 1.$$

$\nabla f = \langle 1, 1 \rangle$

$g(x,y) = x^2 + 4y^2 - 1 = 0, \quad \nabla g = \langle 2x, 8y \rangle$

$1 = \lambda 2x \qquad x = \frac{1}{2\lambda}$

$1 = \lambda 8y \qquad y = \frac{1}{8\lambda}$

$\frac{1}{4\lambda^2} + \frac{4}{64\lambda^2} = 1 \Rightarrow \frac{1}{4\lambda^2} + \frac{1}{16\lambda^2} = 1$

$\frac{5}{16\lambda^2} = 1$

$\lambda^2 = \pm\sqrt{\frac{5}{16}}$

$\lambda = \sqrt{\frac{5}{16}} \qquad x = \frac{1}{2(\sqrt{5}/4)} = \frac{2}{\sqrt{5}} \qquad \left(\frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}}\right)$

$y = \frac{1}{8(\sqrt{5}/4)} = \frac{1}{2\sqrt{5}}$

$\lambda = -\sqrt{\frac{5}{16}} \qquad x = -\frac{1}{2(\sqrt{5}/4)} = -\frac{2}{\sqrt{5}} \qquad \left(-\frac{2}{\sqrt{5}}, -\frac{1}{2\sqrt{5}}\right)$

$y = -\frac{1}{8(\sqrt{5}/4)} = -\frac{1}{2\sqrt{5}}$

$f\left(\frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}}\right) = \frac{2}{\sqrt{5}} + \frac{1}{2\sqrt{5}} = \frac{5}{2\sqrt{5}} = \frac{1}{2}\sqrt{5}$ max

$f\left(-\frac{2}{\sqrt{5}}, -\frac{1}{2\sqrt{5}}\right) = -\frac{1}{2}\sqrt{5}$ min

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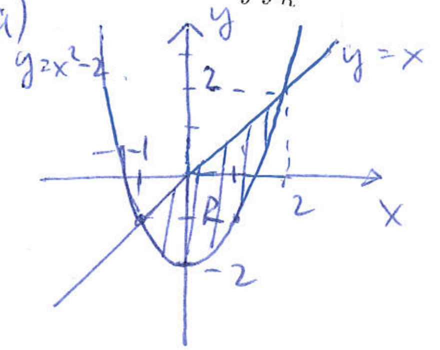
4. **(15 pt)** Let $R$ be the region in the $xy$-plane bounded by $y = x^2 - 2$ and $y = x$.

   (a) Sketch $R$.

   (b) Evaluate $\iint_R 6x \, dA$.

a)




b) the points of the intersection:

$x^2 - 2 = x$

$x^2 - x - 2 = 0$

$D = (-1)^2 - 4 \cdot 1 \cdot (-2) = 9$

$x = \frac{1 \pm \sqrt{9}}{2 \cdot 1}$, $x_1 = \frac{1-3}{2} = -1$

$x_2 = \frac{1+3}{2} = 2$

$R = \{(x,y) \mid -1 \le x \le 2, \ x^2 - 2 \le y \le x\}$

$$\iint_R 6x \, dA = \int_{-1}^{2} \int_{x^2-2}^{x} 6x \, dy \, dx = \int_{-1}^{2} \left( 6xy \Big|_{x^2-2}^{x} \right) dx =$$

$$\int_{-1}^{2} (6x^2 - 6x(x^2-2)) dx = \int_{-1}^{2} (6x^2 - 6x^3 + 12x) dx =$$

$$\left( 6\frac{x^3}{3} - 6\frac{x^4}{4} + 12\frac{x^2}{2} \right) \Big|_{-1}^{2} = 2(8-(-1)) -$$

$$\frac{3}{2}(16-1) + 6(4-1) = 18 - \frac{45}{2} + 18 = \frac{27}{2}$$

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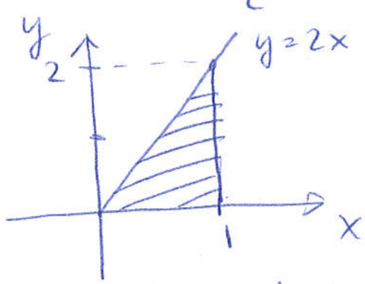
5. **(15 pt)** Consider the integral
$$\int_0^2 \int_{y/2}^1 xy\,dx\,dy.$$

(a) Sketch the region of integration and change the order of integration to $dy\,dx$.

(b) Evaluate the integral. (You may use whichever order of integration you like.)

a) $R = \{(x,y) \mid 0 \le y \le 2, \ \frac{y}{2} \le x \le 1\}$




$x = \frac{y}{2}$

$y = 2x$

$R = \{(x,y) \mid 0 \le x \le 1, \ 0 \le y \le 2x\}$

$\int_0^1 \int_0^{2x} xy\,dy\,dx =$ b) $\int_0^1 \left( x \frac{y^2}{2} \Big|_0^{2x} \right) dx =$

$\int_0^1 x(2x^2 - 0)\,dx = \int_0^1 2x^3\,dx = 2\frac{x^4}{4}\Big|_0^1 =$

$\frac{1}{2}(1-0) = \frac{1}{2}$

or: $\int_0^2 \int_{y/2}^1 xy\,dx\,dy = \int_0^2 \left( y \frac{x^2}{2} \Big|_{y/2}^1 \right) dy =$

$\int_0^2 y \frac{1}{2}\left(1 - \frac{y^2}{4}\right) dy = \frac{1}{2}\int_0^2 \left(y - \frac{y^3}{4}\right) dy =$

$\frac{1}{2}\left(\frac{y^2}{2} - \frac{1}{4}\frac{y^4}{4}\right)\Big|_0^2 = \frac{1}{2}\Big(\frac{1}{2}(4-0) -$

$\frac{1}{16}(16-0)\Big) = \frac{1}{2}(2-1) = \frac{1}{2}$

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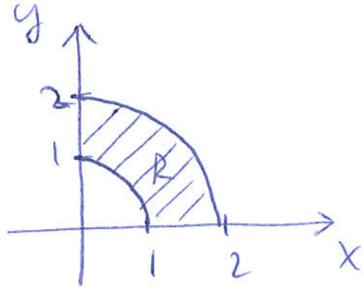
6. **(15 pt)** Let $D = \{(x, y) \mid x \geq 0,\, y \geq 0,\, 1 \leq x^2 + y^2 \leq 4\}$.

   (a) Sketch $D$.

   (b) Express $D$ in polar coordinates.

   (c) Compute $\iint_D \cos(x^2 + y^2)\, dA$.

a)




b) $D = \{(r, \theta) \mid 0 \leq \theta \leq \frac{\pi}{2},\ 1 \leq r \leq 2\}$

c) $\iint_D \cos(x^2+y^2)\,dA = \int_0^{\frac{\pi}{2}} \int_1^2 r\cos r^2\, dr\, d\theta =$

$u = r^2 \quad du = 2r\,dr$, if $r = 1$, then $u = 1$; $r = 2$, $u = 4$

$\frac{1}{2}\int_0^{\frac{\pi}{2}} \int_1^4 \cos u\, du\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\sin u \Big|_1^4\right) d\theta =$

$\frac{1}{2}(\sin 4 - \sin 1)\int_0^{\frac{\pi}{2}} d\theta = \frac{1}{2}(\sin 4 - \sin 1)\,\theta\Big|_0^{\frac{\pi}{2}} =$

$\frac{\pi}{4}(\sin 4 - \sin 1)$

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7. **(10 pt)** Evaluate the integral

$$\int_{-1}^{1}\int_{0}^{1}\int_{y}^{2y}(2z+x)\,dz\,dy\,dx.$$

$$\int_{-1}^{1}\int_{0}^{1}\int_{y}^{2y}(2z+x)\,dz\,dy\,dx = \int_{-1}^{1}\int_{0}^{1}\left(2\,\frac{z^2}{2}+xz\right)\Big|_{y}^{2y}dy\,dx =$$

$$\int_{-1}^{1}\int_{0}^{1}\left(4y^2-y^2+x(2y-y)\right)dy\,dx =$$

$$\int_{-1}^{1}\int_{0}^{1}\left(3y^2+xy\right)dy\,dx = \int_{-1}^{1}\left(3\,\frac{y^3}{3}+x\,\frac{y^2}{2}\right)\Big|_{0}^{1}dx =$$

$$\int_{-1}^{1}\left(1+\frac{x}{2}\right)dx = \left(x+\frac{1}{2}\,\frac{x^2}{2}\right)\Big|_{-1}^{1} =$$

$$(1-(-1))+\frac{1}{4}(1-1) = 2$$