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1. **(15pt)** Consider the function

$$f(x, y) = 3x^2 + 2xy - 2.$$

(a) Find the tangent plane to the graph of $f(x, y)$ at $(1, 2, 5)$.

(b) Use linear approximation and your answer to part a) to estimate $f(1.2, 2.1)$.

a) $f_x = 6x+2y \quad f_y = 2x$

$f_x(1,2) = 6\cdot 1 + 2\cdot 2 = 10$

$f_y(1,2) = 2\cdot 1 = 2$

$z = 10(x-1) + 2(y-2) + 5$

b) $L(x,y) = 10(x-1) + 2(y-2) + 5$

$L(1.2, 2.1) = 10(1.2-1) + 2(2.1-2) + 5 =$

$2 + 0.2 + 5 = 7.2$

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2. **(15 pt)** Find the critical points of the function

$$f(x, y) = x^4 + 2y^2 - 4xy.$$

For each critical point, use the second derivative test to classify it as either a local minimum, a local maximum, or a saddle point.

$f_x = 4x^3 - 4y = 0$ $\quad f_y = 4y - 4x = 0$

$x = y$

$4y^3 - 4y = 0$

$y^3 - y = 0$

$y(y^2 - 1) = 0$

$y = 0$ or $y = \pm 1$

if $y = 0$, $x = 0$

if $y = 1$, $x = 1$

if $y = -1$, $x = -1$

critical points

$(0,0), (1,1), (-1,-1)$

$f_{xx} = 12x^2$ $\quad f_{xy} = -4$ $\quad f_{yy} = 4$

$D(x,y) = 12x^2 \cdot 4 - 16 = 48x^2 - 16$

$D(0,0) = -16$ - saddle point

$D(1,1) = 48 - 16 > 0$ $\quad f_{xx}(1,1) = 12 \cdot 1^2 > 0$, local min

$D(-1,-1) = 48 - 16 > 0$ $\quad f_{xx}(-1,-1) = 12 \cdot (-1)^2 > 0$, local min

3. (**10 pt**) Use Lagrange multipliers to find the absolute maximum and minimum of

$$f(x,y) = x^2 + y^2 - 2x + 4y + 5$$

subject to the constraint

$$x^2 + y^2 = 1$$

Solution: We have: $\nabla f = \langle 2x-2, 2y+4 \rangle$ and $\nabla g = \langle 2x, 2y \rangle$. Therefore, the system is:

$$2x - 2 = \lambda \cdot 2x \quad 2y + 4 = \lambda \cdot 2y \quad x^2 + y^2 = 1$$

from which we get::

$$2x - 2\lambda x = 2 \quad 2y - 2\lambda y = -4$$

or

$$x = \frac{1}{1-\lambda} \quad y = -\frac{2}{1-\lambda}$$

If we plug into the third equation, we get:

$$\frac{4}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 1$$

which can be easily solved to yield $\lambda = 1 \pm \sqrt{5}$.

If $\lambda = 1 - \sqrt{5}$, then $x = \dfrac{1}{1-(1-\sqrt{5})} = \dfrac{1}{\sqrt{5}}$ and $y = -\dfrac{2}{1-(1-\sqrt{5})} = -\dfrac{2}{\sqrt{5}}$

If $\lambda = 1 + \sqrt{5}$, then $x = \dfrac{1}{1-(1+\sqrt{5})} - = \dfrac{1}{\sqrt{5}}$ and $y = -\dfrac{2}{1-(1+\sqrt{5})} = \dfrac{2}{\sqrt{5}}$

The maximum occurs at $\left(\dfrac{1}{\sqrt{5}}, -\dfrac{2}{\sqrt{5}}\right)$ with $f\left(\dfrac{1}{\sqrt{5}}, -\dfrac{2}{\sqrt{5}}\right) = 6 + 2\sqrt{5}$.

The minimum occurs at $\left(-\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right)$ with $f\left(-\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right) = 6 - 2\sqrt{5}$.

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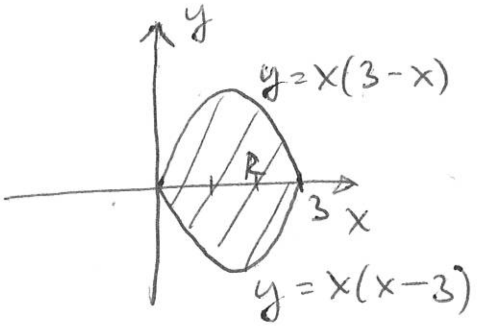
4. **(15 pt)** Compute the integral of $f(x, y) = x$ over the region R bounded by the parabolas

$$y = x(3 - x) \text{ and } y = x(x - 3).$$



$$\iint_R x\,dA = \int_0^3 \int_{x^2-3x}^{3x-x^2} x\,dy\,dx =$$

$$\int_0^3 xy \Big|_{x^2-3x}^{3x-x^2} dx =$$

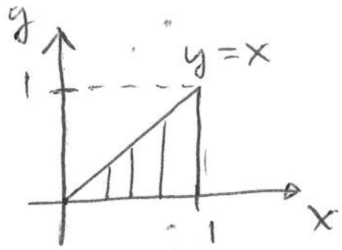
$$\int_0^3 3x^2 - x^3 - (x^3 - 3x^2)\,dx =$$

$$\int_0^3 3x^2 - x^3 - x^3 + 3x^2\,dx = \int_0^3 6x^2 - 2x^3\,dx =$$

$$6\frac{x^3}{3} - 2\frac{x^4}{4}\Big|_0^3 = 2 \cdot 27 - \frac{1}{2} \cdot 81 = \frac{108-81}{2} = \frac{27}{2}$$

5. **(10 pt)** Compute the following integral by changing the order of integration:

$$\int_0^1 \int_y^1 \cos(x^2)\, dx\, dy.$$



$R = \{(x,y) \mid 0 \le y \le 1,\ y \le x \le 1\}$

$R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x\}$

$$\int_0^1 \int_y^1 \cos(x^2)\,dx\,dy = \int_0^1 \int_0^x \cos(x^2)\,dy\,dx =$$

$$\int_0^1 y\cos(x^2)\Big|_0^x dx = \int_0^1 x\cos(x^2)\,dx$$

$u = x^2 \quad du = 2x\,dx$

if $x=0$ then $u=0$

$x=1$ $\quad u=1$

$$\int_0^1 x\cos(x^2)\,dx = \frac{1}{2}\int_0^1 \cos u\,du =$$

$$\frac{1}{2}\sin u\Big|_0^1 = \frac{1}{2}(\sin 1 - \sin 0) =$$

$$\frac{1}{2}\sin 1$$

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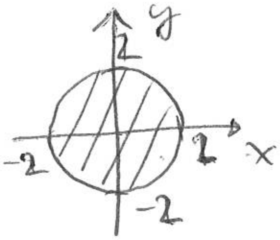
6. **(10 pt)** Compute the double integral $\iint_R e^{x^2+y^2}\,dA$ where R is the disc of radius 2 centered at $(0,0)$.



$$R = \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 2\}$$

$$r^2 = x^2 + y^2$$

$$\iint_R e^{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^{2} r e^{r^2}\,dr\,d\theta =$$

$$u = r^2 \qquad du = 2r\,dr$$

$$\text{if } r = 0 \text{ then } u = 0$$

$$r = 2 \qquad u = 2^2 = 4$$

$$\frac{1}{2}\int_0^{2\pi}\int_0^{4} e^u\,du\,d\theta = \frac{1}{2}\int_0^{2\pi} e^u \Big|_0^4 \,d\theta =$$

$$\frac{1}{2}\int_0^{2\pi} e^4 - e^0\,d\theta = \frac{1}{2}(e^4 - 1)\theta\Big|_0^{2\pi} =$$

$$\pi(e^4 - 1)$$

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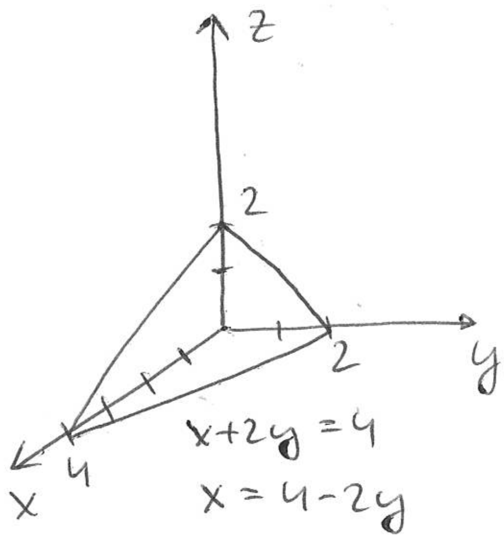
7. **(10pt)** Write down an iterated integral for the triple integral $\iiint_D x\,dV$, where D is the tetrahedron formed by the coordinate planes and the plane $x + 2y + 2z = 4$. **DO NOT** evaluate the integral.

$$\iiint_D x\,dV = \int_0^2 \int_0^{4-2y} \int_0^{\frac{1}{2}(4-x-2y)} x\,dz\,dx\,dy$$



$x + 2y = 4$

$x = 4 - 2y$

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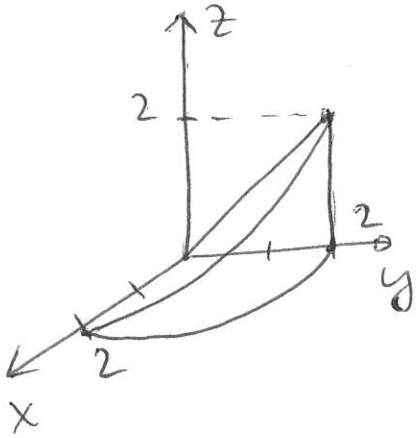
8. **(15 pt)** Compute the integral

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^y x\, dz\, dy\, dx$$

by changing to cylindrical coordinates.



$$D = \{ (x, y, z) \mid 0 \le x \le 2,\ 0 \le y \le \sqrt{4-x^2},\ 0 \le z \le y \}$$

$$y = \sqrt{4-x^2},\quad y \ge 0,\quad x^2 + y^2 = 4$$

$$D = \{ (r, \theta, z) \mid 0 \le \theta \le \frac{\pi}{4},\ 0 \le r \le 2,\ 0 \le z \le r \sin\theta \}$$

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^y x\, dz\, dy\, dx = \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{r\sin\theta} r\cos\theta\, r\, dz\, dr\, d\theta =$$

$$\int_0^{\frac{\pi}{2}} \int_0^2 r^2 \cos\theta\, z \Big|_0^{r\sin\theta} dr\, d\theta =$$

$$\int_0^{\frac{\pi}{2}} \int_0^2 r^3 \cos\theta \sin\theta\, dr\, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin 2\theta \left(\frac{r^4}{4} \Big|_0^2 \right) d\theta =$$

$$\frac{1}{2} \int_0^{\frac{\pi}{2}} \sin 2\theta \left(\frac{16}{4} - 0 \right) d\theta = -\frac{4}{2} \cdot \frac{1}{2} \cos 2\theta \Big|_0^{\frac{\pi}{2}} =$$

$$-\left(\cos 2\frac{\pi}{2} - \cos 0 \right) = 2 \cdot 1 = 2$$