MATHEMATICS 220: FINAL EXAM
University of Illinois at Chicago
(Nicholls, Protsak, Shulman)
December 13, 2018

Please read the exam carefully and follow all instructions. **SHOW ALL OF YOUR WORK.** Please put a box around your final answer.

1. (20 points) Find the general solution of the differential equation
$$\frac{dz}{dt} - \frac{9z}{t} = t^9 \cos(t), \quad t > 0.$$

2. (20 points) Solve the initial value problem
$$\left[\frac{1}{x} + 2xy^3\right] dx + \left[\cos(y) + 3x^2y^2\right] dy = 0, \quad y(e) = 0.$$

3. (25 points) A brine solution of salt flows at a constant rate of 2 L/min into a large tank that initially held 200 L of brine solution in which was dissolved 100 g of salt. The solution inside the tank is kept well stirred and flows out of the tank at the same rate. If the concentration of salt in the brine entering the tank is 1 g/L, determine the mass of salt in the tank after t minutes.

4. (20 points) Solve the initial value problem for $t > 0$,
$$t^2y'' - 2ty' + 2y = 0, \quad y(1) = 2, \quad y'(1) = 0.$$

5. (25 points) Find a general solution to the differential equation using the **Method of Variation of Parameters.** (**Any other method will receive zero points.**)
$$y'' + 4y = \frac{12}{\sin(2t)}, \quad 0 < t < \pi/2.$$

6. (20 points) Calculate the inverse Laplace transform of
$$G(s) = \frac{-s + 17}{s^2 + s - 12}.$$

7. (20 points) Solve the symbolic initial value problem
$$y''(t) + y(t) = \delta(t - \pi/2), \quad y(0) = 0, \quad y'(0) = 1.$$

8. (25 points) Find the general solution for the system of differential equations

$$x'(t) + y(t) = t^2, \quad -x(t) + y'(t) = -1.$$

9. (25 points) Consider the function $f(x) = x - 1$ on the interval $[0, \pi]$.

   (a) (3 points) Sketch the even extension of f on the interval $[-\pi, \pi]$.

   (b) (22 points) Compute the Fourier cosine series of f.

List of Laplace Transforms

1. $\mathcal{L}\{1\} = \dfrac{1}{s}, \quad s > 0$
2. $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}, \quad s > a$
3. $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad s > 0$
4. $\mathcal{L}\{\sin(bt)\} = \dfrac{b}{s^2+b^2}, \quad s > 0$
5. $\mathcal{L}\{\cos(bt)\} = \dfrac{s}{s^2+b^2}, \quad s > 0$
6. $\mathcal{L}\{e^{at}t^n\} = \dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
7. $\mathcal{L}\{e^{at}\sin(bt)\} = \dfrac{b}{(s-a)^2+b^2}, \quad s > a$
8. $\mathcal{L}\{e^{at}\cos(bt)\} = \dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$
9. $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$
10. $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
11. $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
12. $\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$
13. $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \dfrac{d^n}{ds^n}\mathcal{L}\{f\}(s)$
14. $\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)$
15. $\mathcal{L}\{u(t-a)\}(s) = \dfrac{e^{-as}}{s}$
16. $\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$
17. If f has period T then

$$\mathcal{L}\{f\}(s) = \frac{F_T(s)}{1-e^{-sT}} = \frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$$

18. $\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$

List of PDE Formulae

1. The solution of the homogeneous heat equation $u_t = \beta^2 u_{xx}$ with Dirichlet boundary conditions is:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(\beta n\pi/L)^2 t} \sin\left(\frac{n\pi}{L}x\right).$$

2. The solution of the homogeneous heat equation $u_t = \beta^2 u_{xx}$ with Neumann boundary conditions is:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-(\beta n\pi/L)^2 t} \cos\left(\frac{n\pi}{L}x\right).$$

3. The inhomogeneous heat equation has a solution of the form $u(x,t) = v(x) + w(x,t)$, where v is the steady–state solution and w solves a homogeneous heat equation.