Math 180 Calculus 1 Worksheets

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About this booklet

This booklet contains worksheets for the Math 180 Calculus 1 course at the University of Illinois at Chicago.

There are 27 worksheets, each covering a certain topic of the course curriculum. At the end of the booklet there are 2 review worksheets, covering parts of the course (based on a two-midterm model). In a 15-week semester, completing 2 worksheets a week, there should be enough time to complete all the worksheets.

Each worksheet, except the reviews, begins with a list of keywords. These are sorted in the index at the end of the booklet, to make studying and topic-finding easier. The electronic version of this booklet has a hyperlinked index. The calculator icon 🖩 in the margin indicates that a certain question can be done with a calculator. A double exclamation point !! in the margin indicates that a certain question is noticeably more difficult than the others.

To both students and instructors using this booklet - if you find any mistakes, or would like to suggest changes, please send all your comments to `bodem@uic.edu`.

Acknowledgments

This booklet was first organized by Martina Bode, Director of Calculus at UIC, at the end of the Fall 2015 semester. For the first edition Teaching assistants Robert Kozma, Jānis Lazovskis, and Lujia Wang helped put everything together based on worksheets that were created by a team of graduate students. The updates for the second and third editions were put together by Jānis Lazovskis and Martina Bode.

Thanks to all Teaching Assistants that helped to write the initial worksheets and to all that took the time to give feedback.

Some questions, in modified form, are taken from *Calculus: Early Transcendentals* by Briggs, Cochran, and Gillett, the textbook for this course.

First printing: January 2016
Second printing: August 2016
Third printing: August 2017
Fourth printing: January 2018

This version typeset in LaTeX December 9, 2017.

Contents

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1 Functions and slopes

Keywords: *functions, domain, graphing, slope, secant lines*

1. In your own words, describe what is the *domain* and *range* of a function.

2. Consider the following conditions on a function f:

   · the domain of f is all real numbers
   · $f(x) \geqslant 0$ for any real number x
   · $-f(4) = f(-4)$

   Answer the questions below using different functions in each part.

   (a) Give an example of a function that satisfies the first condition.

   (b) Give an example of a function that satisfies the second condition.

   (c) Give an example of a function that satisfies the third condition.

   !! (d) Give an example of a function that satisfies all three conditions.

3. Consider the function $f(x) = \sin(x)$.

   (a) Draw the graph of f from 0 to 4π below. Make sure to label the axes and unit lengths.

   (b) Find the slopes of the secant lines between the given pairs of points below.

   i. $(0, f(0))$ and $(\pi/2, f(\pi/2))$

   ii. $(0, f(0))$ and $(\pi, f(\pi))$

   iii. $(0, f(0))$ and $(3\pi/2, f(3\pi/2))$

   iv. $(0, f(0))$ and $(2\pi, f(2\pi))$

   v. $(0, f(0))$ and $(3\pi, f(3\pi))$

   vi. $(0, f(0))$ and $(4\pi, f(4\pi))$

   (c) Generalize from the above to find the slope of the secant line between $(0, f(0))$ and $(k\pi, f(k\pi))$ for any positive or negative integer $k = 0, 1, 2, 3, \ldots$.

!! (d) Besides the pairs of points in part (c) above, how many other pairs of points exist with the same property? Write down three other such pairs.

2 Properties of limits

Keywords: *limits, piecewise function, secant lines, average velocity, instantaneous velocity*

1. Consider the piecewise defined function:

$$f(x) = \begin{cases} 3 & \text{if } x < 0, \\ \sin(x) + 1 & \text{if } 0 \leqslant x \leqslant \pi, \\ x & \text{if } \pi < x < 5, \\ (x-6)^2 & \text{if } x \geqslant 5. \end{cases}$$

   You may assume that $\lim\limits_{x \to 0} \sin x = 0$ and $\lim\limits_{x \to \pi} \sin x = \sin \pi$.

   (a) For each of $c = 0, \pi, 4, 5$, when does $\lim\limits_{x \to c} f(x)$ exist and when does it not exist? Show all your work.

!! (b) Make all the limits in part (a) exist by adding different numbers to different parts of the function.

2. Consider the position function $p(t) = t^2 - t$, where t is time in seconds. Complete the following table with the appropriate average velocities.

Time interval	$[1, 2]$	$[1, 1.5]$	$[1, 1.1]$	$[1, 1.01]$
Average velocity				

   Take a guess as to what the instantaneous velocity will be at $t = 1$.

3. On Monday afternoon Olive gets on the Kennedy Expressway at North Avenue and gets off at Taylor Street, driving a distance of 3 miles on the highway. There is a 60 MPH sign at the Randolph Street exit, which is 2 miles down from the North Avenue ramp.

   (a) On Tuesday she gets a speeding ticket, because video cameras on the highway indicated that she was on the highway for 2 minutes and 30 seconds. If the recording is to be trusted, what was her average speed over the 3 mile section?

   (b) On Wednesday Olive goes to court and claims it took her 1 minute and 20 seconds to drive the first two miles, where the speed limit was 90 MPH, and 1 minute and 10 seconds to drive the third mile. If Olive is to be trusted, what was her average speed

      i. over the first two miles?

      ii. over the last mile?

   (c) On Thursday the county judge decides that Olive deserves the ticket if her speed is more than 60 MPH when she passes the sign. Is Olive's argument enough to prove that her speed was not more than 60 MPH when she passed the sign? Why or why not?

3 Computing limits

Keywords: *limits, graphing, one-sided limits*

1. Let f, g be two functions with $\lim_{x \to c} f(x) = 2$ and $\lim_{x \to c} g(x) = 6$. Showing all your steps, simplify the following limits.

   (a) $\lim_{x \to c} [8g(x)]$

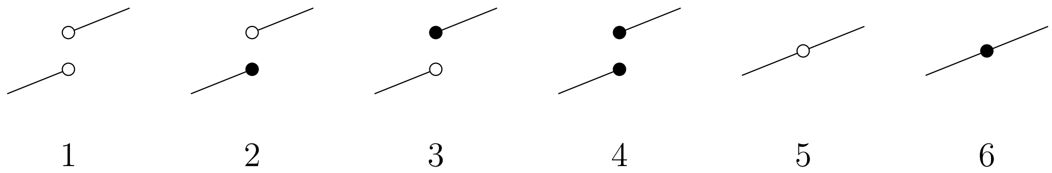
   (b) $\lim_{x \to c} [5f(x) + 9g(x)]$

   (c) $\lim_{x \to c} \left[\dfrac{g(x)}{22} \right]$

   (d) $\lim_{z \to c} [f(z)/3 - z]$

!! (e) $\lim_{x \to c+2} [2f(x - 2)]$

2. Below are six graphs.



Which of these graphs satisfy the properties below?

(a) the graph is a function
(b) the graph is defined at every point
(c) left-hand-side limits exist at every point
(d) right-hand-side limits exist at every point
(e) limits exist at every point
(f) the graph is a function f and $\lim_{x \to a^-} f(x) = f(a)$ for all points a
(g) the graph is a function f and $\lim_{x \to a^+} f(x) = f(a)$ for all points a

3. Consider the function $f(x) = \begin{cases} \cos(x), & \text{if } x \leqslant 0, \\ x^2, & \text{if } 0 < x \leqslant 2, \\ 2x, & \text{if } x > 2. \end{cases}$

(a) Draw the graph of $f(x)$ below. Make sure to label the axes and unit lengths.

(b) Do $\lim_{x \to 0} f(x)$ and $\lim_{x \to 2} f(x)$ exist? Justify your answer by computing the 1-sided limits.

4. Compute the following limits or state that they do not exist.

   (a) $\displaystyle\lim_{x\to0^+}\frac{|x|}{x}$ (b) $\displaystyle\lim_{x\to0^-}\frac{|x|}{x}$ (c) $\displaystyle\lim_{x\to0}\frac{|x|}{x}$

5. With help from the question above, draw a graph of $\frac{|x|}{x}$ on the interval $[-5,5]$. Make sure to label the axes and unit lengths.

4 Computing more limits

Keywords: *limits, squeeze theorem, infinite limits*

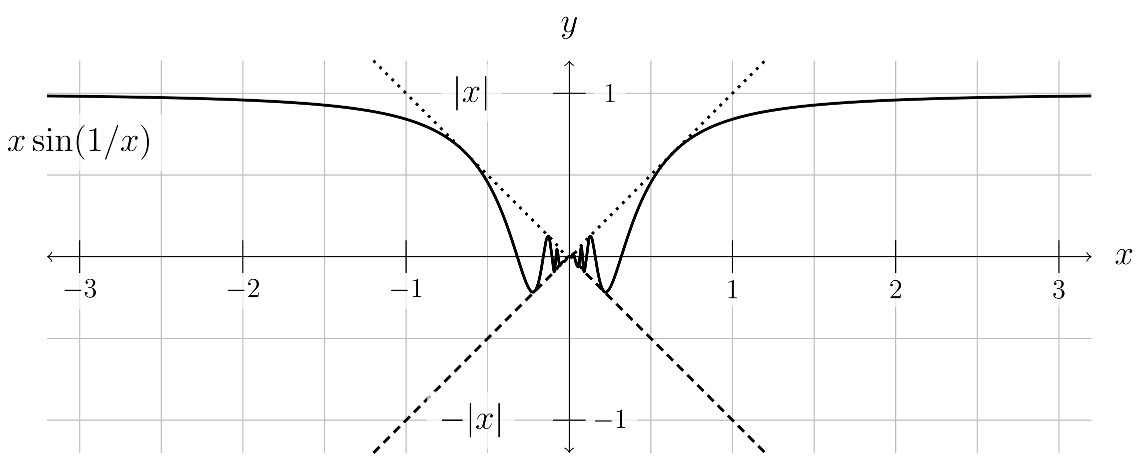
1. Determine the following limits or state that they do not exist.

   (a) $\lim_{x\to 1}\left[\frac{(x-1)(x-2)}{(x-1)^3}\right]$

   (b) $\lim_{x\to 2}\left[\frac{x^3-5x^2+6x}{x^3-4x}\right]$

   (c) $\lim_{x\to 1}\left[\frac{x-1}{\sqrt{x}-1}\right]$

2. Consider the graphs of $y = x\sin(1/x)$, $y = |x|$, and $y = -|x|$, given below.



   Using the graph and the squeeze theorem, evaluate $\lim_{x\to 0}\left[x\sin\left(\frac{1}{x}\right)\right]$.

3. Let $f(x) = x^4 \cos\left(\dfrac{1}{x^4}\right)$.

   (a) Fill in the blanks by using algebraic operations to change the first inequality below.

$$\begin{aligned} -1 \leq &\quad \cos(\theta) &\leq 1 \\ \underline{\qquad} \leq &\quad \cos(1/x^4) &\leq \underline{\qquad} \\ \underline{\qquad} \leq &\ x^4\cos(1/x^4) &\leq \underline{\qquad} \end{aligned}$$

   (b) Apply the squeeze theorem to evaluate $\displaystyle\lim_{x\to 0}\left[x^4\cos\left(\frac{1}{x^4}\right)\right]$.

4. Consider the function $f(x) = \dfrac{1}{x-3}$.

   (a) Evaluate the following limits.

      i. $\displaystyle\lim_{x\to 3^+} f(x)$ ii. $\displaystyle\lim_{x\to 3^-} f(x)$ iii. $\displaystyle\lim_{x\to 3} f(x)$

   (b) Sketch a graph of f below. Make sure to label the axes and unit lengths.

   (c) Explain, in your own words, what is happening at $x = 3$.

5 Limits at infinity and asymptotes

Keywords: *limits, limits at infinity, asmyptotes, graphing*

1. (a) Evaluate the following limits.

   i. $\lim_{x\to-\infty} \left[-2x^4 - x^2 - 8x\right]$

   iii. $\lim_{x\to-\infty} \left[\dfrac{-6x^7 - 4x^2 + 2}{x^2 - 3x + 5}\right]$

   ii. $\lim_{x\to+\infty} \left[\dfrac{3x^5 - x^3 + 8x}{-5x^5 - 7}\right]$

   iv. $\lim_{x\to+\infty} \left[\dfrac{2x^2 - 3x}{x^4 - 7}\right]$

   (b) Do any of the functions in the limits above contain horizontal asymptotes? If so, explain why.

2. Consider the function $f(x) = \dfrac{3x^2 - x}{x^2 - 6x + 5}$.

   (a) List any vertical asymptotes of $f(x)$, and explain, using limits, why each is a vertical asymptote.

   (b) List any horizontal asymptotes of $f(x)$, and explain, using limits, why each is a horizontal asymptote.

3. Consider the function $f(x) = \ln(x-2)$.

   (a) What is the domain of f?

   (b) Graph f below. Make sure to label the axes and unit lengths.

   (c) List all the vertical and horizontal asymptotes of f, and explain, using limits, why each is either a horizontal or vertical asymptote.

6 Continuity

Keywords: *continuity, graphing, intermediate value theorem*

1. On the grid below, draw a function f that is not continuous at $x = 2$ with $\lim_{x \to 2} f(x) = 1$.

2. On the grid below, draw a function f that is defined everywhere and whose limit $\lim_{x \to 2} f(x)$ is undefined.

3. Are the functions you drew above continuous at $x = 2$

   (a) from the left?

   (b) from the right?

Recall that a function is *continuous from the left* at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$, and *continuous from the right* if $\lim_{x \to a^+} f(x) = f(a)$.

4. On the grid below, draw the function

$$f(x) = \begin{cases} 5 & \text{if } x < 0, \\ \sin(\pi x) & \text{if } 0 \leqslant x \leqslant 1, \\ x - 1 & \text{if } x > 1. \end{cases}$$

On what intervals is $f(x)$ continuous? Explain your reasoning!

5. Determine whether or not f is continuous at $x = a$. If not, explain why.

   (a) $f(x) = \sqrt{x - 5}$ at $a = 3$

   (b) $f(x) = \dfrac{x^2 - 4}{x + 2}$ at $a = -2$

6. Recall the intermediate value theorem (IVT).

   (a) What are the conditions of the IVT?

   (b) What is the conclusion of the IVT?

   (c) Use the IVT to determine if $x^3 + 2x + 1$ has a root on the interval $(-1, 0)$.

7. Does the equation $x + e^x = 0$ have a solution on the interval $x \in (-1, 0)$? Why or why not?

7 Definition of the derivative

Keywords: *limit definition of the derivative, secant lines*

1. (a) Write down the limit definition of the derivative of a function f at the point x.

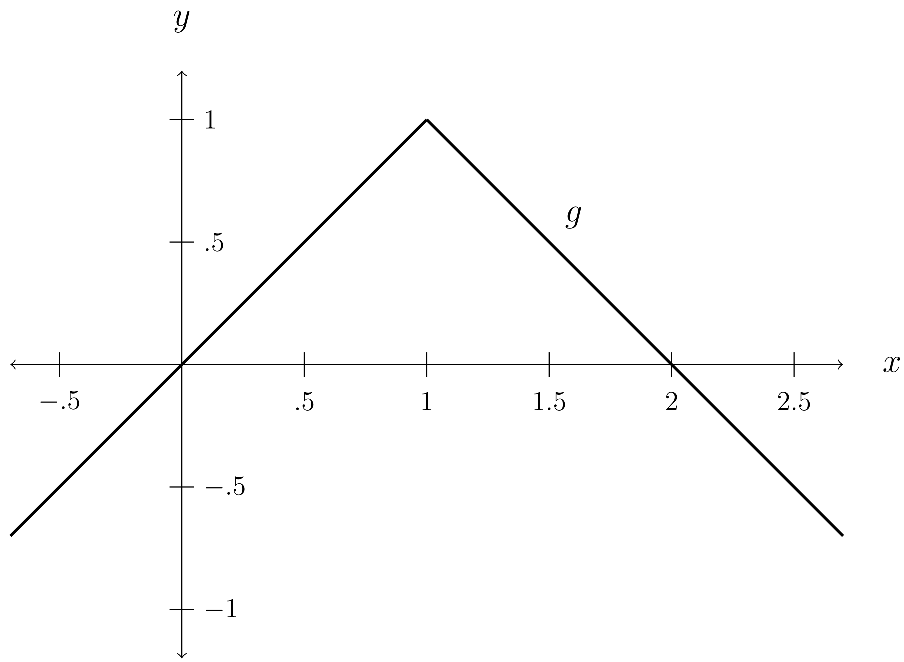
   (b) Using the definition you gave above, compute $f'(x)$ in each of the following cases.

      i. $f(x) = 3x + 5$

      ii. $f(x) = x^2 + 1$

      iii. $f(x) = \frac{2}{x}$

2. The graph of $g(x)$ is shown below.



Sketch the graph of the derivative $g'(x)$ and explain your answer.

3. Consider the function $f(x) = \frac{2}{x}$ from the exercise above.

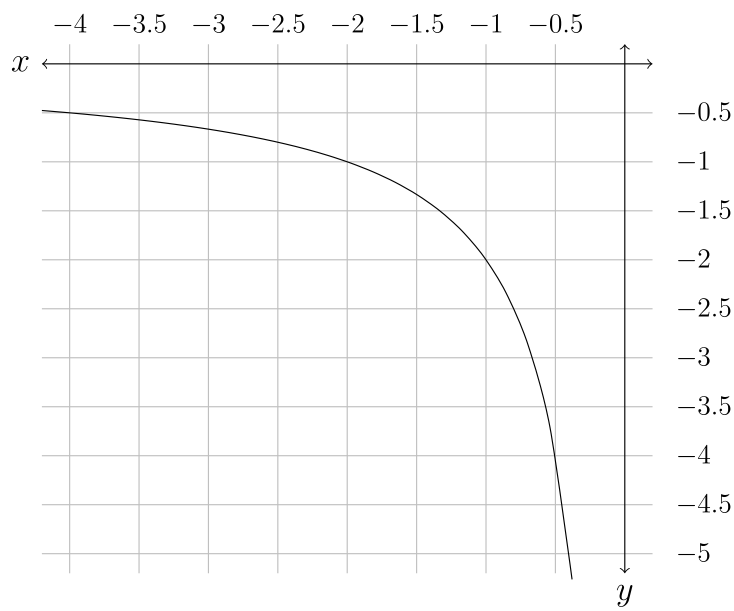
   (a) Use your answer from above to evaluate $f'(-2)$.

   (b) Complete the following table of difference quotients.

h	1.5	1	0.5	0.1
$\dfrac{f(-2+h)-f(-2)}{h}$				

   Do the quotients get closer to $f'(-2)$ as h gets smaller?

   (c) For each value of h in the table above, draw a line through $(-2, f(-2))$ and $(-2+h, f(-2+h))$ on the figure below.



   The slopes of these lines should correspond to the values of the difference quotients in the table.

   (d) Draw the line tangent to the graph at $x = -2$ and estimate the slope. How does this estimate compare to your computed value of $f'(-2)$ above?

8 Computing derivatives

Keywords: *derivatives, differentiation, power rule, tangent lines*

1. Find the derivative of the following functions.

   (a) $h(t) = t$

   (e) $f(w) = 2w^3 + 3w^2 + 100$

   (b) $g(t) = \pi e^3$

   (f) $s(t) = 4\sqrt{t} - \frac{1}{4}t^4 + t + 1$

   (c) $f(x) = 5x^6$

   (g) $h(x) = (x^2 + 1)^2$

   (d) $g(y) = 6\sqrt{y}$

   (h) $g(w) = \dfrac{w^3 - w}{w}$

2. Consider the function $f(x) = 2e^x - 6x$.

   (a) Find $f'(x)$. What does $f'(1)$ mean geometrically?

   (b) Find an equation of the tangent line of f at $x = 1$.

   (c) Find the point(s) on the graph of f where the tangent line is horizontal.

   !! (d) Find the equation(s) of the horizontal tangent lines of f.

3. Find the first 3 derivatives of the following functions. What are the 4th, 5th, 6th, nth derivatives?

   (a) $f(x) = 3x^2 + 5e^x$

   (b) $g(x) = 3x^3 + 5x^2 + 6x$

!! 4. Find the following limit by identifying it with the definition of a derivative:

$$\lim_{h \to 0} \frac{(1+h)^8 + (1+h)^3 - 2}{h}$$

9 Product and quotient rules, derivatives of trigonometric functions

Keywords: *product rule, quotient rule, derivatives, trigonometric functions*

1. Compute the derivatives of the following functions using the product and quotient rules. Do not simplify!

   (a) $f(x) = (4x^2 + 2x + 3)(\sqrt{x} + 2)$

   (b) $f(x) = \dfrac{4x^2 + 2x + 3}{\sqrt{x} + 2}$

2. Use the product and quotient rules and the fact that $\dfrac{d}{dx}\big(\sin x\big) = \cos x$ and $\dfrac{d}{dx}\big(\cos x\big) = -\sin x$ to show that the following equalities are true.

   (a) $\dfrac{d}{dx} \sec x = \sec x \tan x$

   (b) $\dfrac{d}{dx} \cot x = -\csc^2 x$

3. Assume that f and g are differentiable functions about which we know very little. In fact, assume that all we know about these functions is the following table of data:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	5	9	9	-3
1	3	-3	2	6
2	-5	3	8	5

(a) Let $h(x) = f(x)g(x)$, what is $h'(0)$?

(b) Let $i(x) = (2\sin x + 3)g(x)$, what is $i'(0)$?

(c) Let $j(x) = \dfrac{f(x)}{g(x)}$, what is $j'(1)$?

(d) Let $k(x) = \dfrac{1}{f(x)}$, what is $k'(2)$?

!! 4. Let $f(v)$ be the gas consumption (in liters / km) of a car going at velocity v (in km / hr). Suppose $f(80) = \frac{5}{100}$ and $f'(80) = \frac{5}{10\ 000}$.

(a) Let $g(v)$ be the distance the car goes on one liter of gas at velocity. v. Find $g(80)$ and $g'(80)$.

(b) Let $h(v)$ be the gas consumption in liters per hour. Find $h(80)$ and $h'(80)$.

(c) What is the practical meaning of these quantities?

10 The chain rule, and derivatives as rate of change

Keywords: *chain rule, meaning of the derivative, position, velocity, acceleration, average and marginal cost*

1. Compute the derivatives of the following functions using the chain rule. Do not simplify!

   (a) $f(x) = (4x^2 + 2x + 3)^9$

   (b) $f(x) = \sin(x^3 + 2)$

   (c) $f(x) = \tan(\sqrt{x})$

   (d) $f(x) = \sec(5x)$

   (e) $f(x) = \cos\left((2x + 1)^3\right)$

2. Given $f(3) = 3$, $f'(3) = 5$, $f(9) = 2$, $f'(9) = 8$.
   If $h(x) = f(\sqrt{x})$, find $h'(9)$.

3. A stone is thrown in the air from a cliff 96 ft above a river. The position of the stone (measured as the height above the river) after t seconds is $s(t) = -16t^2 + 64t + 80$.

   (a) Find the velocity function $v(t)$.

   (b) Find the acceleration function $a(t)$.

   (c) i. Find the time at which the velocity is zero.

   ii. What is happening at that time?

   iii. What is the height of the stone at that time?

   (d) When does the stone hit the ground?

4. A manufacturer produces rolls of fabric with a fixed width. The quantity q, in yards, of fabric sold is a function of the selling price p, in dollars per yard, so we may write $q = f(p)$. Give a sentence in English explaining the following equations.

   (a) $f(20) = 10\,000$

   (b) $f'(20) = -350$

   (c) The revenue is given by $R(q) = pq = pf(p)$. Do you have enough information to calculate $R'(20)$?

11 Derivatives of logarithmic, exponential, and inverse trigonometric functions

Keywords: *differentiation, logarithmic differentiation, inverse functions, trigonometric functions, exponential functions, logarithmic functions*

1. (a) Evaluate $\log_4 8$ and $\ln e^3$.

   (b) Find the exact value of $\log_{10} 0.25 + \log_{10} 0.4$.
   Hint: turn decimals into fractions!

   (c) Solve the equation $2^{x+5} = 3 \cdot 5^x$ for x.

   (d) Evaluate

      i. $\arccos(1)$

      ii. $\arcsin(0)$

      iii. $\arcsin(1/2)$

      iv. $\arctan(-1)$

2. Evaluate the derivatives of the following functions.

   (a) $f(x) = \sin^{-1}(e^{2x})$

   (b) $f(w) = \sin^{-1}(2^w)$

   (c) $f(x) = \tan^{-1}(\log_2 x)$

   (d) $f(x) = \ln\left(\dfrac{x+1}{x-1}\right)$

3. Use logarithmic differentiation to find the derivatives of the following functions.

   (a) $f(x) = (\sin x)^x$

   (b) $f(x) = \left(1 + \dfrac{1}{x}\right)^{2x}$

!! (c) $f(x) = \dfrac{x^{2x}(ex)}{x^{10x+15}} \cdot e^{1+2x}$

12 Related rates

Keywords: *implicit differentation, related rates, word problems*

1. For each equation below, find the equation of the tangent line to the curve at the given point.

   (a) $x^2 + xy + y^2 = 7$ at $(2, 1)$

   (b) $(x + y)^{2/3} = y$ at $(4, 4)$

2. A spherical balloon is inflated and its volume increases at a rate of 15 in^3/min. What is the rate of change of its radius when the radius is 10 in?

3. A 13-foot ladder is leaning against a vertical wall when Jack begins pulling the foot of the ladder away from the wall at a rate of 0.5 ft/s. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 5 ft from the wall?

4. A baseball diamond is a square with sides 90 ft long. A batter is at bat, with runners at first and second base. At the moment the ball is hit, the runner at first base runs to second base at 25 ft/s. Simultaneously, the runner on second base runs to third base at 15 ft/s. How fast is the distance between these two runners changing 2 s after the ball is hit?

5. A kite 40 ft above the ground moves horizontally at a constant speed of 10 ft/s, with a child, holding the ball of kite string, standing motionless on the ground. At what rate is the child releasing the string when

   (a) 50 ft of the string is out?
   (b) 100 ft of the string is out?

13 Maxima and minima, what the derivative tells us

Keywords: *extrema, word problems, monotonicity, concavity, inflection point*

1. Find the global maximum value and the global minimum value of the function $f(x) = x^4 - 2x^2 + 3$ on $[-1, 2]$.

2. Find the absolute maximum and minimum of $f(x) = (x^2 - 1)^3$ on

   (a) $[0, 2]$,

   (b) $[0, 3]$.

3. Let $f(x) = x^3 - 3x$.

   (a) Find the intervals on which $f(x)$ is increasing and on which it is decreasing.

   (b) Find the intervals on which $f(x)$ is concave up and on which it is concave down. Does f have any points of inflection?

4. Let $f(x) = e^{x^3-3x}$.

   (a) Find the intervals on which $f(x)$ is increasing and on which it is decreasing.

   (b) Find the absolute maximum and minimum of $f(x)$ on the interval $[0, 2]$.

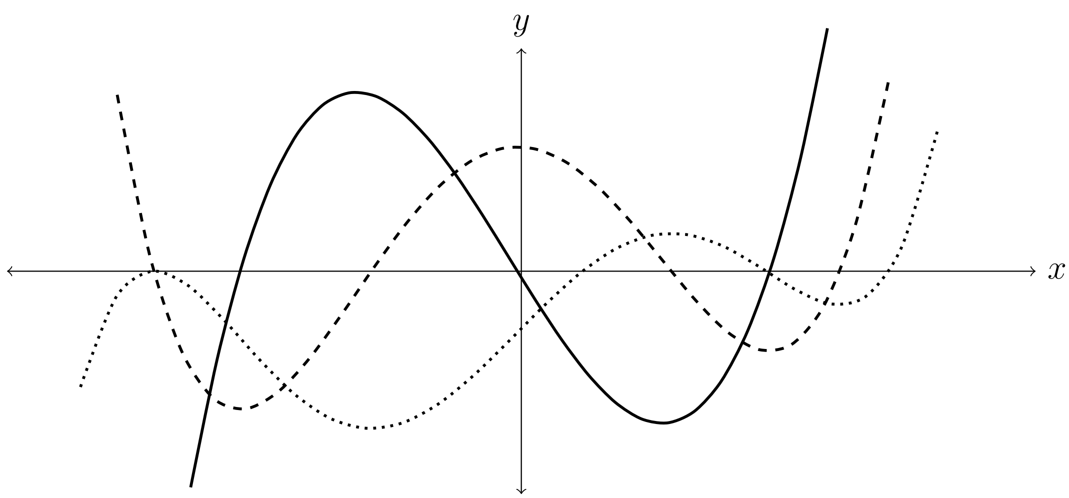
14 Properties of graphs of functions

Keywords: *monotonicity, concavity, inflection point*

1. (a) What is an inflection point?

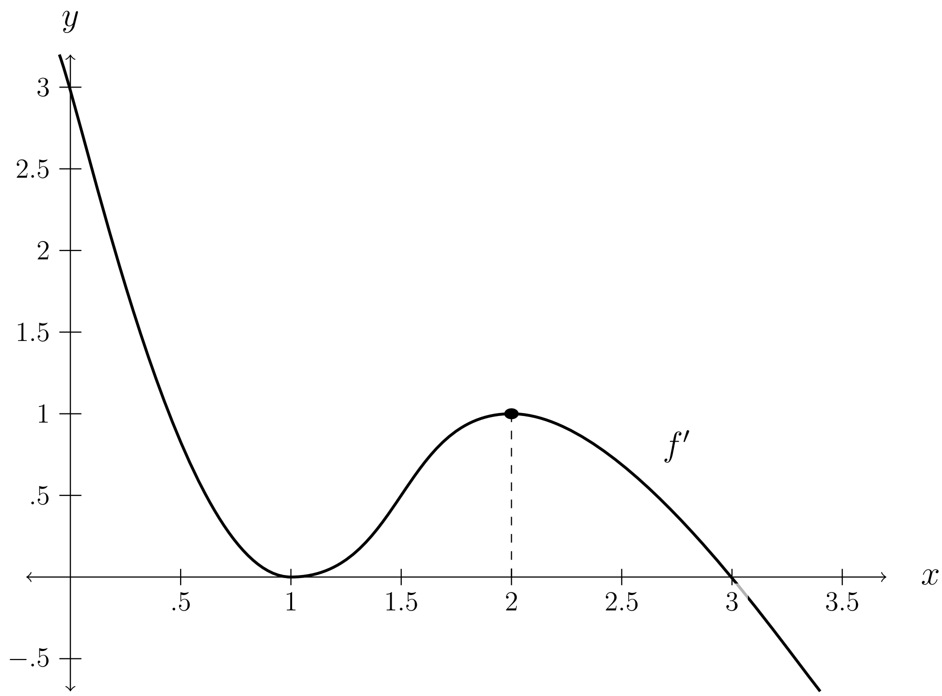
   (b) Give an example of a function that does not have an inflection point at $x = 0$, but satisfies $f''(0) = 0$. Sketch its graph below.

2. The following figure shows the graphs of f, f', and f''.



   Which curve is which? Justify your answers.

3. The graph of the derivative f' of a function $f(x)$ is shown below.



(a) On what intervals is f increasing or decreasing?

(b) At what values of x does f have a local maximum or minimum?

(c) On what intervals is f concave upward or downward? You may assume an inflection point occurs exactly between the two shown extrema.

(d) Assuming that $f(0) = 0$, sketch a graph of f.

4. Consider the function $f(x) = 2x^3 + 3x^2 - 12x + 1$.

   (a) Locate the critical points of f.

   (b) Use the first derivative test to locate the local maxima and local minima of f.

   (c) Use the second derivative test to locate the points of inflection, and compare your answers with part (b).

   (d) Identify the absolute minimum and maximum values of f on the interval [-2,4].

!! 5. Find the intervals of concavity of $f(x) = e^{\sin(x)}$.

15 Graphs of functions and introduction to optimization

Keywords: *graphing, optimization, word problems*

1. Let $f(x) = \dfrac{10x^3}{x^2-1}$. Assume $f'(x) = \dfrac{10x^2(x^2-3)}{(x^2-1)^2}$ and $f''(x) = \dfrac{20x(x^2+3)}{(x^2-1)^3}$.

   (a) Does f have any vertical or horizontal asymptotes?

(b) On what intervals is f increasing or decreasing?

(c) At what values of x does f have a local maximum or minimum?

(d) On what intervals is f concave upward or downward?

(e) Sketch a graph of f.

2. In your own words, describe what is the goal of an optimization problem, without using the word optimize.

3. A three sided fence is to be built next to a straight river which forms the fourth side of a rectangular region. The enclosed area is to equal 3200 ft^2. The cost of the fence is \$2/ft. This question will have you find the dimensions that minimize the cost of the fence.

   (a) What quantity needs to be optimized?

   (b) Draw a picture modeling this situation.

   (c) Find a function for the quantity that is being optimized and find its domain.

   (d) Find the dimensions that minimize the cost of the fence.

16 Optimization

Keywords: *optimization, word problems*

For the problems below, include the domain of the function to be optimized, as well as an argument why your critical point(s) yields an absolute maximum or minimum.

1. What two non-negative real numbers a and b whose sum is 23 maximize $a^2 + b^2$? Minimize $a^2 + b^2$? Solve both questions using optimization techniques even if you know the answer.

2. A square-based, box-shaped shipping crate is designed to have a volume of 16 ft^3. The material used to make the base costs twice as much (per square foot) as the material used in the sides, and the material used to make the top costs half as much (per square foot) as the material in the sides. What are the dimensions of the crate that minimize the cost of the materials?

!! 3. A piece of wire of length 60 cm is cut, and the resulting two pieces are formed to make a circle and a square. Where should the wire be cut to:

(a) minimize the combined area of the circle and the square?

(b) maximize the combined area of the circle and the square?

17 The mean value theorem and introduction to l'Hôpital's rule

Keywords: *mean value theorem, secant lines, word problems,L'Hôpital's rule*

Recall the mean value theorem (MVT), which states: If f is continuous on closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

1. Use this theorem to answer the following questions.

   (a) What is the slope of the secant line passing through $(a, f(a))$ and $(b, f(b))$?

   (b) What is the slope of the tangent line at c?

   (c) What does the MVT say about the two quantities above?

   (d) Rewrite the theorem in terms of instantaneous speed (*hint: tangent line*) and average speed (*hint: secant line*) for a trip between times a and b.

   (e) Is the theorem applicable to the function $f(x) = |x - 1|$ on the interval $[0, 2]$? Why or why not?

2. Suppose we know that $f(x)$ is continuous and differentiable on $[2, 14]$, with $f(2) = -5$ and $f'(x) \leqslant 10$. What is the largest possible value for $f(14)$? Justify your reasoning.

!! 3. A state patrol officer observed a car start from rest at a highway on-ramp. She radioed ahead to a patrol officer standing along the highway 30 miles ahead. When the car reached the location of the second officer 28 minutes later, the driver of the car was given a ticket for exceeding the 60 mph speed limit. Why could the officer conclude that the driver was driving too fast?

4. Compute the following limits using L'Hôpital's rule. Indicate what type of indeterminate limit is given.

   (a) $\lim_{x \to 0} \left[\frac{3 \sin x}{7x} \right]$

   (b) $\lim_{x \to 0^+} \left[\frac{3 \cos x}{7x} \right]$

   (c) $\lim_{x \to 1} \left[\frac{1 - x}{1 - \ln x} \right]$

   (d) $\lim_{x \to 0} \left[\frac{x}{\sin(2x) - \sin(3x)} \right]$

   (e) $\lim_{x \to \infty} \left[\frac{3x^2 + 4x + 5}{6x^2 + 2} \right]$

18 L'Hôpital's rule and antiderivatives

Keywords: *L'Hôpital's rule, indeterminate forms, antiderivaties, initial value problems*

1. Compute the following limits. Think about the cases when it is quicker not to use L'Hôpital's rule.

   (a) $\lim_{x \to e} \left[\frac{\ln(x) - 1}{x - e} \right]$

   (b) $\lim_{x \to 0} \left[\frac{1}{\sin x} - \frac{1}{x} \right]$

   (c) $\lim_{x \to 0^+} \left[x^2 \ln(2x) \right]$

   !! (d) $\lim_{x \to \infty} \left(xe^{1/x} - x \right)$

2. List five types of indeterminate forms that can be evaluated using l'Hôpital's rule.

3. Find the following limits, or argue why they do not exist.

   (a) $\lim_{x \to 0} \left[\cos(x)^{1/x^2} \right]$

!! (b) $\lim_{x \to \infty} \left[\left(1 + \frac{a}{x} \right)^x \right]$

4. Find antiderivatives of the following functions.

   (a) $f(x) = 3x^2 + 2$

   (b) $g(x) = x^4 + 6x^2 + 4x + 1$

   (c) $h(x) = \sec^2\theta + \sec(2\theta)\tan(2\theta)$

5. Find a function G satisfying $G'(x) = x^5 - 2x^{-2} + 1$ and $G(1) = 0$.

6. Astronaut Bob is standing on a platform 3 meters above the moon's surface and throws a rock directly upward with an initial velocity of 16 m/s. The acceleration due to gravity on the moon's surface is 1.6 m/s^2.

   (a) After how many seconds does the rock reach its highest point?

   (b) After how many seconds does the rock fall back onto the moon surface? (Assume the platform and the astronaut have moved away after throwing the rock.)

19 Riemann sums

Keywords: *Riemann sums, area under a curve, graphing, sigma notation*

1. Let $f(x) = \cos(x)$ on $[0, 3\pi/2]$ with $n = 6$.

   - Sketch the graph of the function on the given interval.
   - Calculate Δx and the grid points $x_0, x_1, \ldots, x_n$.
   - Illustrate the n-th left and right Riemann sums.

2. Let $f(x) = x^2 - 1$ on $[0, 4]$ with $n = 4$.

   - Sketch the graph of the function on the given interval.
   - Calculate Δx and the grid points $x_0, x_1, \ldots, x_n$.
   - Illustrate the n-th left and right Riemann sums, and determine which Riemann sum underestimates and which sum overestimates the area under the curve.
   - Calculate the left and right n-th Riemann sums.

3. Repeat the steps in the question above, but with the midpoint Riemann sums instead.

(a) $f(x) = \sqrt{x}$ on $[1, 3]$ with $n = 4$

(b) $f(x) = \frac{1}{x}$ on $[1, 6]$ with $n = 5$

4. Express the following sums using sigma notation:

   (a) $1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5}$

   (b) $1^2 + 2^2 + 3^2 + 4^2 + \cdots + 20^2$

   (c) $-1 + 2 - 3 + 4 - 5 + \cdots - 19$

!! (d) $\dfrac{1}{2} + \dfrac{8}{9} + \dfrac{18}{16} + \dfrac{32}{25} + \dfrac{50}{36} + \dfrac{72}{49}$

20 Sigma notation and more integration

Keywords: *Riemann sums, sigma notation, area under a curve*

1. Write the definition of a definite integral $\int_a^b f(x)dx$ as a limit of Riemann sums.

2. Express the limit $\lim_{n\to\infty}\left[\sum_{k=1}^{n}\sqrt{1+\frac{2k}{n}}\cdot\ln\left(1+\frac{2k}{n}\right)\cdot\frac{2}{n}\right]$ as a definite integral.

3. Suppose $\int_1^4 f(x)dx = 8$ and $\int_1^6 f(x)dx = 5$. Evaluate the following integrals.

   (a) $\displaystyle\int_4^1 -3f(x)dx$

   (b) $\displaystyle\int_4^4 5f(x)dx$

   (c) $\displaystyle\int_6^4 2f(x)dx$

4. Consider the function $f(x) = 8 - 2x$.

   (a) Sketch the graph of f over $[0, 6]$ below. Make sure to label the axes and unit lengths.

   (b) Evaluate $\int_0^6 8 - 2x \, dx$ using geometry.

   That is, draw the graph of the integrand and find the area of the shape representing the integral - do *not* use Riemann sums, guessing, or techniques not yet introduced.

!! (c) This question will have you evaluate $\int_0^6 8 - 2x\ dx$ using the *definition of the integral* as a limit of Riemann sums.

i. Divide the interval $[0,6]$ into n subintervals of equal length Δx, and find the following values:

A. $\Delta x =$

B. $x_0 =$

C. $x_1 =$

D. $x_2 =$

E. $x_3 =$

F. $x_i =$

ii. A. What is $f(x)$? Evaluate $f(x_i)$ for arbitrary i.

B. Rewrite $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ using the information above.

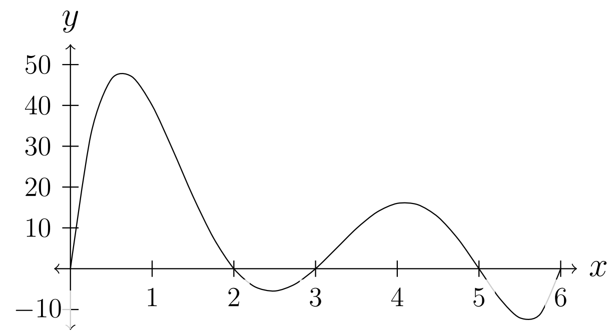
C. Evaluate first the sum, then the limit from the previous part. You may find the following summation formulas useful:

$$\sum_{i=1}^{n} c = c{\cdot}n, \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

21 The fundamental theorem of calculus

Keywords: *fundamental theorem of calculus, integrals, area under a curve, evaluation theorem*

1. Consider the graph below of a function f on the interval $[0, 6]$.



The area function of f is $A(x) = \int_0^x f(t)\, dt$.

(a) Identify the intervals on which A is increasing and on which it is decreasing.

(b) Use this information to identify any local extrema of A and indicate whether each is a local max or local min.

(c) Identify all points at which A is zero.

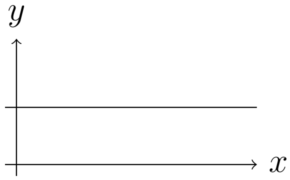
2. Identify whether the following statements are true or false. If true, explain why. If false, give a counterexample.

(a) Suppose that f is a continuous function on $x \geqslant 0$, increasing on $x > 0$. Then the area function $A(x) = \int_0^x f(t)\, dt$ is an increasing function of x.

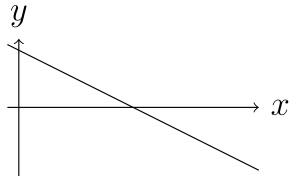
(b) Suppose that f is a continuous positive function on $x \geqslant 0$, decreasing on $x > 0$. Then the area function $A(x) = \int_0^x f(t)\, dt$ is an increasing function of x.

3. For each function in the first row, match it to its area function in the second row.

Functions



A



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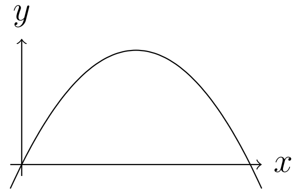

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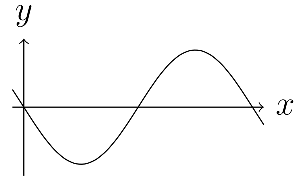
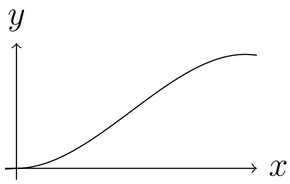
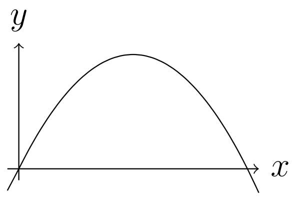


D

Area functions



1



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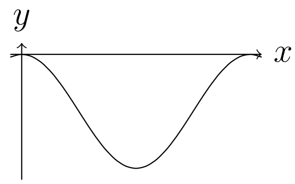

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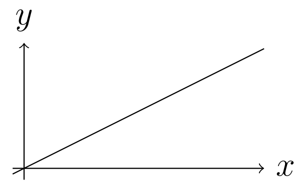


4

4. The Fundamental Theorem of Calculus states that $A'(x) = f(x)$ for the area function $A(x) = \int_0^x f(t)\,dt$. Evaluate the following derivatives.

(a) $\displaystyle \frac{d}{dx}\left(\int_1^x \frac{\sin(t)\,dt}{t}\right)$

(c) $\displaystyle \frac{d}{dx}\left(\int_1^{x^3} \frac{\sin(t)\,dt}{t}\right)$

(b) $\displaystyle \frac{d}{dz}\left(\int_1^z \frac{\sin(t)\,dt}{t}\right)$

(d) $\displaystyle \frac{d}{dx}\left(\int_{x^2}^1 \frac{\sin(t)\,dt}{t}\right)$

5. Evaluate the following integrals using the fundamental theorem of calculus part 1, also known as the *evaluation theorem.*

(a) $\displaystyle\int_0^{\pi/4} \sec^2(\theta)\ d\theta$

(c) $\displaystyle\int_0^{1} x + \sqrt{x}\ dx$

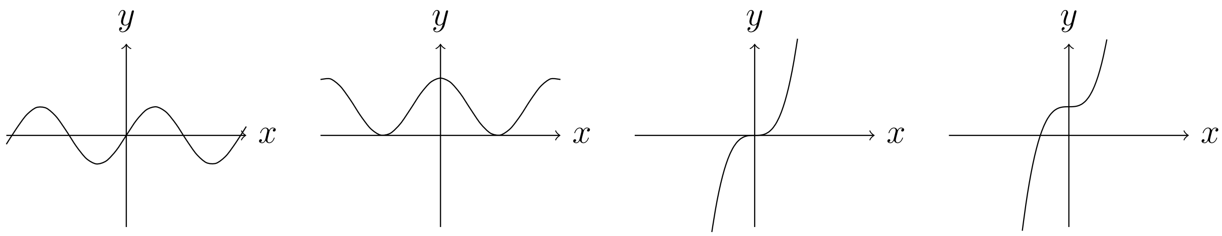
(b) $\displaystyle\int_0^{4} x^3 - 6x^2 + 8x\ dx$

(d) $\displaystyle\int_0^{1} 10e^{2x}\ dx$

22 Properties of the definite integral

Keywords: *integrals of even and odd functions, mean value theorem*

1. Determine if each of the following functions is even, odd, or neither.



2. Complete the sentences below by circling the correct answer.

   (a) If $f(x)$ is even then $f(-x) =$

   i. x ii. $-x$ iii. $-f(x)$ iv. $f(x)$

   (b) If $f(x)$ is odd then $f(-x) =$

   i. x ii. $-x$ iii. $-f(x)$ iv. $f(x)$

   (c) If $f(x)$ is constant then $f(-x) =$

   i. x ii. $-x$ iii. $-f(x)$ iv. $f(x)$

3. Evaluate the following integrals using symmetry arguments.

   (a) $\displaystyle\int_{-3}^{3} x^6 + 2x^4 \; dx$

   (b) $\displaystyle\int_{-10}^{10} \sin(x^3) \; dx$

   (c) $\displaystyle\int_{-\pi/2}^{\pi/2} (\cos(x) + 2\sin(x)) \; dx$

   (d) $\displaystyle\int_{-\pi}^{\pi} \sin(x)\cos(x) \; dx$

4. (a) State the mean value theorem (MVT) for integrals.

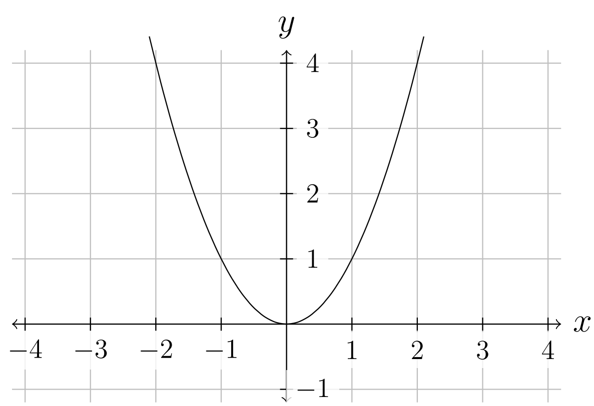
   (b) Use the MVT for integrals to show that there is a point c in the interval $(0, \frac{\pi}{4})$ such that $\sec^2(c) = \frac{4}{\pi}$.

5. Consider the function $f(x) = x^2$.

   (a) What is the average value of f on the interval $[0, 2]$?

   (b) What is the average value of f on the interval $[0, a]$ for any positive number a?

   (c) Let k be the answer to part (a) above. Draw the line $g(x) = k$ on top of the graph of f below.

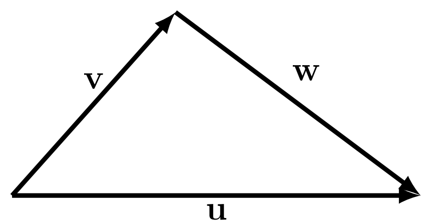


   (d) Between 0 and 2, how do the areas under f and under g compare?

23 Vectors in the plane

Keywords: *vectors, scalar multiplication, vector addition, unit vectors, force vectors*

1. In the picture below, the three vectors form a triangle, The vectors $\mathbf{u}$ and $\mathbf{v}$ are vectors with magnitude 6 and 3.



   (a) Sketch the vectors $2\mathbf{u}$ and $-\frac{1}{3}\mathbf{u}$. What are the magnitudes of these vectors?

   (b) Express $\mathbf{w}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

   (c) Express $\mathbf{u}$ in terms of $\mathbf{v}$ and $\mathbf{w}$.

2. Find a unit vector that has the same direction as the vector $\langle 4, -3 \rangle$.

3. A woman in a canoe leaves the shore and paddles due west at 4 mph in a current that flows northwest at 2 mph. Find the speed and direction of the canoe relative to the shore where she left.

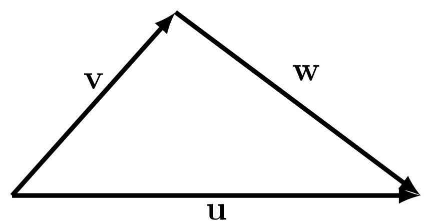
4. Hänsel and Gretel are pulling a sled full of candy up a ramp that is at an angle of $10°$ above the horizontal. Hänsel and Gretel are pulling the sled with a combined force of $20\sqrt{3}$ N at an angle of $40°$ above the horizontal. If the ramp is 4 m long, find a formula for the force vector.

5. Which has a greater horizontal component, a 110 N force directed at an angle of 60^o above the horizontal, or a 60 N force directed at an angle of 30^o above the horizontal?

24 Dot product in the plane

Keywords: *vectors, dot products, angles, projections, vector components, force vectors, work*

1. In the picture below, the three vectors form a triangle, The vectors $\mathbf{u}$ and $\mathbf{v}$ are vectors with magnitude 6 and 3, respectively, and $\mathbf{u} \cdot \mathbf{v} = 12$.



   (a) Find $\mathbf{u} \cdot \mathbf{w}$.

   (b) Find $\mathbf{v} \cdot \mathbf{w}$.

   (c) What is the magnitude of $\mathbf{w}$?

   (d) Sketch the vector $\text{proj}_{\mathbf{u}}\mathbf{v}$.

   (e) Sketch the vector $\text{proj}_{\mathbf{u}}\mathbf{w}$.

2. Recall the question from the previous worksheet:

   Hänsel and Gretel are pulling a sled full of candy up a ramp that is at an angle of 10° *above the horizontal. Hänsel and Gretel are pulling the sled with a combined force of* $20\sqrt{3}$ *N at an angle of* 40° *above the horizontal.*

   If the ramp is 4 m long, how much work is done by Hänsel and Gretel in pulling the sled from the bottom of the ramp to the top?

3. (a) Find the cosine of the angle between the two vectors $\langle 2, 3\rangle$ and $-\mathbf{i} + \mathbf{j}$.

   (b) Find the projection of the vector $\langle 2, 3\rangle$ along the vector $\langle -1, 1\rangle$.

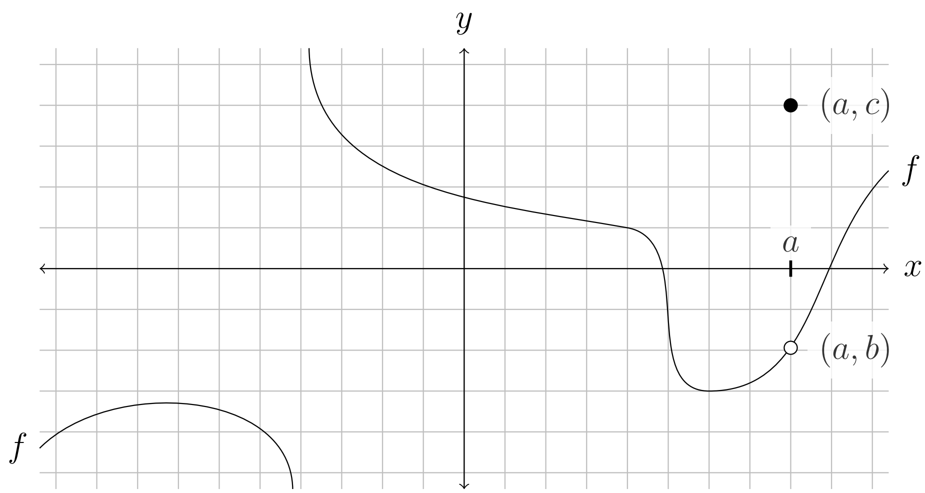
4. Suppose $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors.

   (a) Use the dot product to show that $\mathbf{u}$ is orthogonal (that is, perpendicular) to $\mathbf{v} - \mathrm{proj}_{\mathbf{u}}\mathbf{v}$.

   (b) Draw a picture to show that the two vectors $\mathbf{v} - \mathrm{proj}_{\mathbf{u}}\mathbf{v}$ and $\mathbf{u}$ are perpendicular.

1 Review of worksheets 1-10

1. Consider the following graph of a function f.

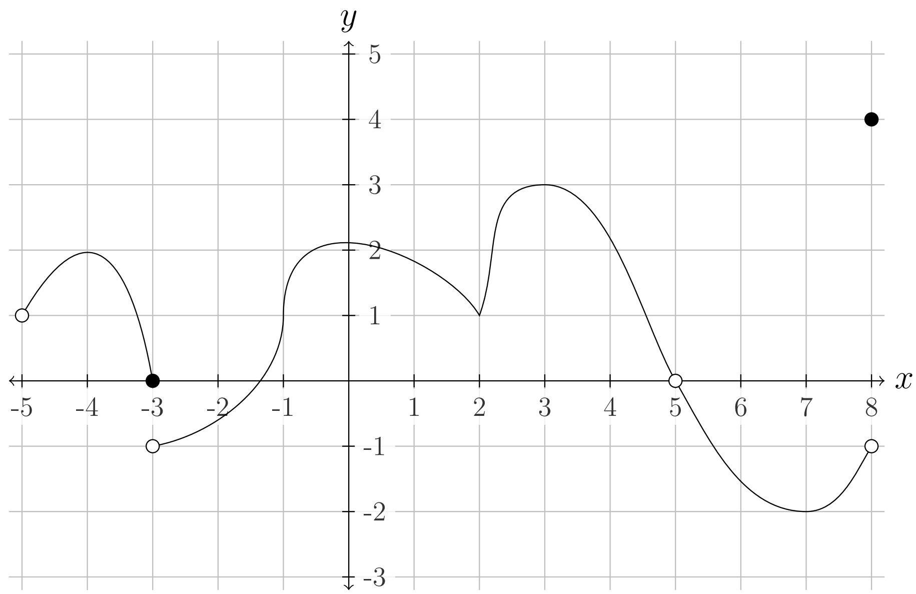


(a) What is $\lim_{x\to a} f(x)$?

(b) What is $f(a)$?

(c) Is f continuous at a? Justify your answer.

2. Consider the graph of the function f below.



For this problem, and only for this problem, you do not need to show work or provide explanations.

(a) Evaluate the following limits. If the limit does not exist state so.

i. $\lim_{x\to -3^+} f(x)$ ii. $\lim_{x\to -3} f(x)$ iii. $\lim_{x\to 5} f(x)$

(b) Give all intervals on which $f(x)$ is continuous.

(c) For which values of x in the interval $(-5, 8)$ is $f(x)$ not differentiable?

(d) For which values of x is $f'(x) = 0$?

3. Evaluate the following limits. If the limit does not exist, explain why it does not exist.

   (a) $\displaystyle\lim_{x\to 4} \frac{x^2-6x+8}{x^2-x-12}$

   (b) $\displaystyle\lim_{x\to 3} \frac{x-3}{2-\sqrt{x+1}}$

   (c) $\displaystyle\lim_{x\to 4} \frac{x^2-x-12}{|x-4|}$

4. Differentiate the following functions. You do not need to simplify your answers.

   (a) $f(x) = \pi x^3 + \pi x + \pi e^x + e^2$

   (b) $g(x) = (3 + x^2)\tan(x)$

   (c) $h(x) = \dfrac{\sin x}{4x + 1}$

   (d) $s(y) = \cos(5y)$

   (e) $g(s) = e^{s^2}$

   (f) $f(x) = \sqrt{\sin x}$

   (g) $g(x) = x\tan(xe^x)$

   (h) $h(x) = \sqrt{x + \sqrt{x}}$

   (i) $f(x) = \sin^2(x^5)$

!! (j) $t(z) = \sin(\cos(\tan(2z + 1)))$

5. Let $f(x) = \dfrac{x^3}{x^3 - 4x}$.

   (a) Find all the vertical and horizontal asymptotes of f.

   (b) For the vertical asymptotes $x = a$, find $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$.

   (c) For horizontal asymptotes, find $\lim\limits_{x \to +\infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$.

6. If $y = f(x)$ satisfies $f(2) = 5$ and $f'(2) = 3$, find the equation of the tangent line to f at $x = 2$.

7. Suppose that $f(x) = (g(x))^3$, where both f and g are differentiable functions. If $g(0) = 2$ and $g'(0) = 1/2$, compute the equation of the tangent line to $f(x)$ at $x = 0$.

8. Use the limit definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{2}{x+1}$.

9. Use the squeeze theorem to evaluate $\lim_{x \to 1} \left[(x-1)^2 \sin\left(\dfrac{1}{(x-1)^2} \right) \right]$.

10. Use the intermediate value theorem to show that $f(x) = x^3 + 4x - 1$ has a root between 0 and 1.

11. The cost of extracting T tons of ore from a copper mine is $C = f(T)$ dollars. What does it mean to say that $f'(2000) = 100$?

2 Review of worksheets 11-16

1. Suppose you are trying to find the value of $\sqrt{75}$.

   (a) Find the linear approximation to $f(x) = \sqrt{x}$ at $x = 64$ and $x = 81$.

   (b) Estimate the value of $\sqrt{75}$ using these two linear approximations.

   (c) Find the value of $\sqrt{75}$ using a caluclator. Which of the two approximations above was closer?

2. Find the equation for the tangent line at the point $(1, 1)$ of the curve $x^3 + 4xy - y^3 = 4$.

3. The cost of extracting T tons of ore from a copper mine is $C = f(t)$ dollars. What does it mean to say that $f'(2000) = 100$?

4. Evaluate the following derivatives, using logarithmic differentiation where appropriate.

   (a) $\dfrac{d}{dx}\left((x^3+2)e^{3x}\right)$

   (b) $\dfrac{d}{d\theta}\left(\tan^{-1}(\theta^2+1)\right)$

   (c) $\dfrac{d}{dx}\left(x^{\tan(x)}\right)$

   (d) $\dfrac{d}{dx}\left(\ln(\sin(x^2+1))\right)$

!! (e) $\dfrac{d}{dx}\left(\dfrac{(x+1)^{3/2}\sqrt{x^2-1}}{(x^2+x+1)^3}\right)$

5. Find the absolute maximum value and the absolute minimum value of the function $f(x) = x - 3x^{1/3}$ on $[0, 8]$.

6. Find all the local maxima and the local minima of $g(x) = 300 - 8x^2 + 4x^3 + x^4$.

7. Let $f(x) = e^{-x^2}$.

   (a) Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

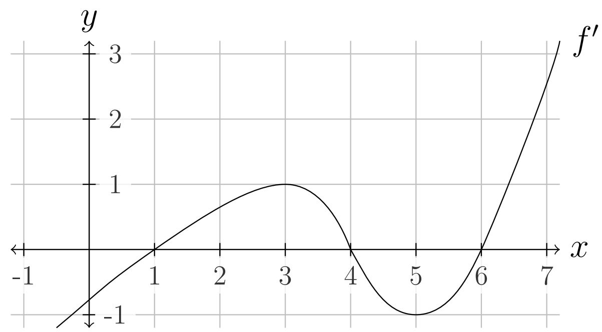
   (b) On what intervals is f increasing? decreasing?

   (c) At what values of x does f have a local maximum? global maximum? Does f have a local minimum?

   (d) On what intervals is f concave upward? On what intervals is it concave downward? What are the x-coordinates of the inflection points of f?

   (e) Sketch the graph of the function.

8. The graph of the derivative function f' of a function f is shown below.



(a) On what intervals is f increasing?

(b) At what values of x does f have a local maximum? At what values does it have a local minimum?

(c) On what intervals is f concave upward? On what intervals is it concave downward?

(d) What are the x-coordinates of the inflection points of f?

(e) If it is known that $f(0) = 0$, sketch a possible graph of f. Make sure to label all axes and unit lengths.

9. Let $f(x) = -x^3 - 2x^2 + x + 1$ and $g(x) = \ln(x+1) + 1$.

   (a) Find the equation of the tangent line to f at $x = 0$

   (b) Show that g has the same linear approximation as f at $x = 0$.

10. Let $f(x) = x^3 + ax^2 + bx + c$ where a, b, and c are constants with $a \geqslant 0$ and $b > 0$.

    (a) Over what intervals is f concave up? Concave down?

    (b) Show that f must have exactly one inflection point.

    (c) Given that $(0, -2)$ is the inflection point of f, compute a and c and then show that f has no critical points.

11. A dock is 6 feet above the water. Suppose you stand on the edge of the dock and pull a rope attached to a boat at a constant rate of 2 ft/s. Assume that the boat remains at water level while you are pulling. At what speed is the boat approaching the dock when

    (a) the boat is is 20 feet from the dock? (b) the boat is 10 feet from the dock?

12. Suppose that a bear 6 feet tall wobbles with a ground speed of 2 ft/s towards a search light that is at the top of a 15-foot pole. How fast is the tip of bear's shadow moving along the ground when the bear is 25 feet from the light?

3 Review of worksheets 1-24

1. A function defined for all x has the following properties: f is increasing, f is concave down, $f(5) = 3$ and $f'(5) = 1/2$.

   (a) Sketch a possible graph for $f(x)$ below. Make sure to label all axes and unit lengths.

   (b) Is it possible that $f'(1) = 1/4$? Why or why not? Write up a solid argument to support your claim. Is there a theorem that can be used to support your claim?

2. Assuming $4 \leqslant f(x) \leqslant 5$ for all x, evaluate $\lim\limits_{x\to 0} xf(x)$.

3. Let $f(x) = \begin{cases} cx+1 & \text{if } x < 1, \\ x^2 - x - 3 & \text{if } x \geqslant 1. \end{cases}$

   (a) Find $\lim_{x\to 1^-} f(x)$, and $\lim_{x\to 1^+} f(x)$.

   (b) Find the value of c which makes $f(x)$ continuous at $x = 1$.

4. Find values of c and d that make the following function continuous and differentiable everywhere.

$$f(x) = \begin{cases} cx^2 + 3x + 4, & \text{if } x < 1, \\ 7x + d, & \text{if } x \geqslant 1. \end{cases}$$

5. Use the limit definition to compute the derivative function $f'(x)$ for the function $f(x) = \frac{3}{x}$.

6. Find the following limits. Show all your work and use L'Hôpital's rule where appropriate!

   (a) $\lim\limits_{x \to 0^+} \left[\dfrac{\cos x}{\ln x} \right]$

   (b) $\lim\limits_{x \to \infty} \left[\dfrac{x^2 + 2}{2x^2 + 1} \right]$

   (c) $\lim\limits_{x \to 0} \left[\dfrac{x^5}{x^5 + 1} \right]$

   (d) $\lim\limits_{x \to \infty} \left[\dfrac{e^x}{1 + e^x} \right]$

   (e) $\lim\limits_{t \to 0} \left[\dfrac{\ln(t+1)}{3^t - 2^t} \right]$

   (f) $\lim\limits_{x \to \infty} \left[x^{(e^{-x})} \right]$

7. Let f be a continuous, differentiable function with $f(0) = 0$ and $-1 \leqslant f'(x) \leqslant 5$ for all $x \in [-3, 3]$.

   (a) Can $f(3)$ ever be negative? Give a reason for your answer.

   (b) Can $f(-3)$ ever be negative? Give a reason for your answer.

   (c) How large can $f(3)$ be? How large can $f(-3)$ be?

   (d) Must f have a critical point between -3 and 3? Either show your reasoning or give a counterexample.

8. Differentiate the following functions, you do not need to simplify your answers.

   (a) $f(x) = \sqrt[5]{x^8}$

   (b) $f(x) = \ln(x^2 - 4)$

   (c) $f(t) = \arctan\left(\dfrac{2}{t^2}\right)$

   (d) $f(x) = x^2 \tan(2x^3 + 1)$

   (e) $f(x) = x^{3x}$

9. Use the Intermediate Value Theorem to show that the polynomial $f(x) = x^2 + x - 1$ has a zero in the interval $(0, 1)$.

10. Find the shortest distance from the point $(4, 0)$ to a point on the parabola $y^2 = 2x$. *Hint: Minimize the distance squared.*

11. Consider the function $f(x) = x^{1/3}$.

(a) Determine the absolute extrema of f on $[-1, 8]$.

(b) Find the intervals where f is increasing and where it is decreasing.

(c) Find the intervals where f is concave upward and where it is concave downward.

(d) Sketch the graph of f on the grid below.

(e) Why does the Mean Value Theorem fail for f on $[-8, 8]$?

12. Let $f(x) = \dfrac{x^2 - x - 2}{x^2 - 2x + 1} = \dfrac{x^2 - x - 2}{(x-1)^2}$, $f'(x) = \dfrac{5 - x}{(x-1)^3}$ and $f''(x) = \dfrac{2(x-7)}{(x-1)^4}$.

   (a) Find the following limits.

      i. $\lim\limits_{x \to \infty} f(x)$

      ii. $\lim\limits_{x \to -\infty} f(x)$

      iii. $\lim\limits_{x \to 1^-} f(x)$

      iv. $\lim\limits_{x \to 1^+} f(x)$

   (b) Does f have any asymptotes?

   (c) On what intervals is f increasing? decreasing? At what values of x does f have a local maximum? local minimum?

(d) On what intervals is f concave upward? concave downward? Does f have any points of inflection?

(e) Sketch a graph of f using information from the previous parts.

(f) Does f have an absolute maximum value? absolute minimum value?

13. Find the linear approximation $L(x)$ of the function $f(x) = \ln(x)$ at $a = 1$ and use it to approximate $\ln(9/10)$.

14. Find the equation for the tangent line at the point $(1, -1)$ of the curve given implicitly by:
$$x^2y - 3y^3 = x^2 + 1.$$

15. Suppose f is a function that is continuous and differentiable everywhere with the following properties:

    - f has a horizontal asymptote at $y = 1$
    - $f'(x) > 0$ on the interval $(-\infty, -2)$ and $(0, \infty)$, and $f'(x) < 0$ on the interval $(-2, 0)$
    - $f''(x) > 0$ on the interval $(-\infty, -3)$ and $(-1, \infty)$, and $f''(x) < 0$ on the interval $(-3, -1)$
    - $f(-3) = 2$, $f(-2) = 3$, and $f(-1) = 2$

    Sketch the graph of f on the grid below.

16. A paper cup shaped like a cone with height 20 cm and top radius 4 cm springs a leak and water begins to pour out through the bottom. If the water is flowing out at a rate of 2 cm^3/second when the water level is at the halfway point, how fast is the water level decreasing at this time?

17. Consider the integral
$$\int_{-4}^{4} 1 - x^2 \; dx.$$
Use a right Riemann sum approximation with 4 subintervals to estimate the value of this integral.

18. If $\displaystyle\int_{-4}^{5} f(t)\,dt = 4$ and $\displaystyle\int_{1}^{5} f(t)\,dt = -3$, evaluate $\displaystyle\int_{1}^{-4} \big(f(t) + 2t + 1\big)\,dt$.

19. Let $F(x) = \displaystyle\int_{\ln x}^{0} t^2 \cos(t)\, dt$. Find $F'(x)$.

20. Consider the function $F(x) = \displaystyle\int_0^x |t-1| - 2\, dt$ defined on the interval $(0, 5]$.

    (a) Identify the intervals on which F is increasing and on which it is decreasing.

    (b) Identify the intervals on which F is concave up and on which it is concave down

    (c) Sketch the graph of F on the grid below. Make sure to label all axes and unit lengths.

21. A person pushing a lawnmower exerts a force of 30 pounds in the direction of the handle, which makes an angle of 30^o with the ground. How much work is done in moving the lawnmower 20 feet?

22. (a) Find the vector projection of $\langle 3, 4\rangle$ onto the positive x-axis.

    (b) Find the angle between these two directions.

23. Debby uses a constant force of magnitude 10 N to haul a package from the point $P = (3, 2)$ to the point $Q = (7, 5)$. If Debby pulls at an angle of $60°$ relative to the vector $\vec{PQ}$, how much work does she do in moving the package from P to Q?

24. An aquarium acquires the clairvoyant celebrity octopus Paul, and needs to house him in a tank with a volume of 30 000 ft^3. If Paul is kept in a tank with a square base, he has the ability to predict the outcome of the next presidential election. The sides of the tank are to be made of glass, and the bottom of the tank will be made of acrylic. The top of the tank will remain open. Glass costs \$50 / ft^2, and acrylic costs \$10 / ft^2. Because acquiring Paul was expensive, the aquarium needs to minimize the cost of the tank. Which dimensions minimize the cost of the tank, and how much will the tank cost?

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