Math 181 Calculus II
Worksheet Booklet

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About this Booklet

This booklet contains worksheets for the Math 181 Calculus II course at the University of Illinois at Chicago.

There are 28 worksheets, each covering a certain topic of the course curriculum. In a 15-week semester, completing 2 worksheets a week, there should be enough time to complete all the worksheets.

Each worksheet, except the reviews, begins with a list of keywords. These are sorted in the index at the end of the booklet, to make studying and topic-finding easier. The electronic version of this booklet has a hyperlinked index. The calculator icon 🖩 in the margin indicates that a certain question can be done with a calculator. A double exclamation point !! in the margin indicates that a certain question is noticeably more difficult than the others.

To both students and instructors using this booklet - if you find any mistakes, or would like to suggest changes, please send all your comments to `bodem@uic.edu`.

Acknowledgments

This booklet was first organized by Martina Bode, Director of Calculus at UIC, at the end of the Spring 2016 semester. Robert Kozma, Jānis Lazovskis, and John Lesieutre helped put everything together for the first edition. The updates for the second and third editions were put together by Matthew Fitzpatrick, Robert Kozma, and Jānis Lazovskis.

Some questions, in modified form, are taken from *Calculus: Early Transcendentals* by Briggs, Cochran, and Gillett, the textbook for this course.

First printing: August 2018
This version typeset in LaTeX July 19, 2018.

Contents

Weekly Worksheets **1**

1 Review 1: Definite integrals ... 1

2 Review 2: Fundamental theorem of calculus ... 4

3 Working with integrals ... 7

4 Substitution ... 10

5 Regions between curves and volumes ... 13

6 Volumes by slicing ... 16

7 Length of curves and work ... 19

8 Integration by parts ... 23

9 More integration by parts and trigonometric integrals ... 26

10 Partial fractions and improper integrals ... 29

11 Introduction to sequences ... 33

12 Introduction to series ... 36

13 The divergence, p-series, and ratio tests ... 39

14 The comparison test ... 42

15 The alternating series test and approximating functions with polynomials ... 45

16 Power series ... 49

17 Numerical integration ... 53

18 Taylor series ... 56

19 Parametric equations ... 59

20 Polar coordinates 62

21 Linear systems 66

22 Matrices 69

23 More matrices 72

24 Linear Maps 76

Review Worksheets **80**

1 Review for first midterm 80

2 Review for second midterm 86

3 Review for final exam (part one) 91

4 Review for final exam (part two) 96

Index **101**

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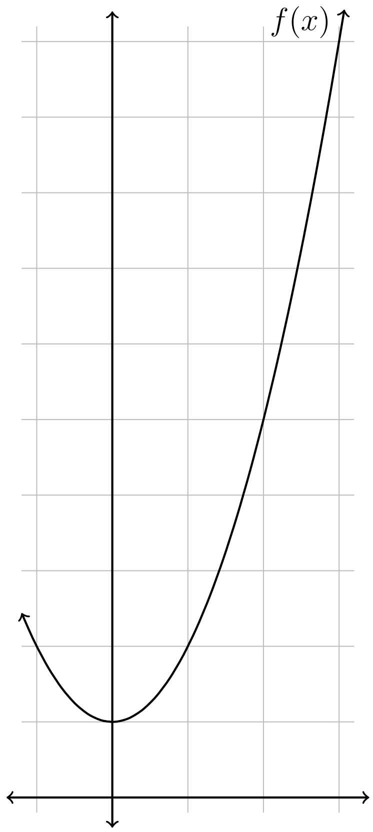
1 Review 1: Definite integrals

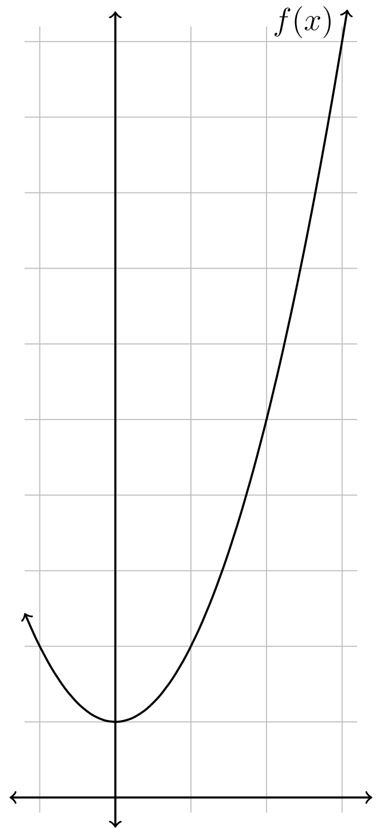
Keywords: *integration, definite integral, Riemann sum, area under curve*

1. Illustrate and evaluate the following Riemann sums for $f(x) = 1 + x^2$ on the interval $[-1, 3]$ with n equally spaced subintervals by first calculating Δx and the grid points $x_0, x_1, \ldots, x_n$.

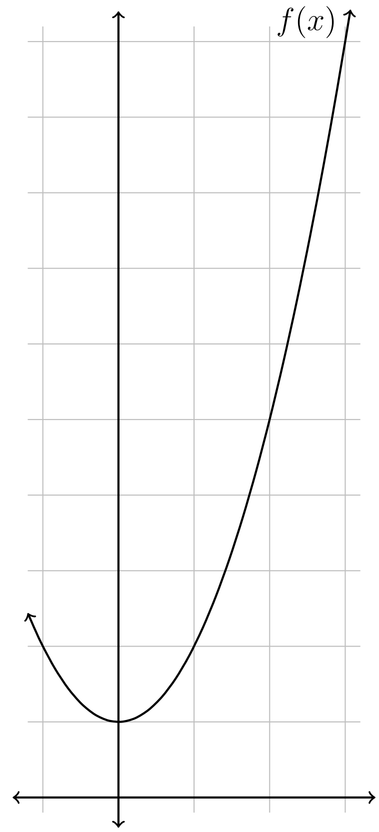
   (a) A left Riemann sum with $n = 4$.

   (b) A right Riemann sum with $n = 4$.

   (c) A midpoint Riemann sum with $n = 2$.

2. (a) For each subinterval in the left and right Riemann sums above, determine if the corresponding Riemann rectangle is an underestimate or overestimate of the area under the curve.

   (b) In general, when is a left/right Riemann sum an overestimate/underestimate?

3. Write the definition of a definite integral $\int_a^b f(x)\,dx$ as a limit of Riemann sums.

4. Set up a definite integral which represents the area under the curve $f(x) = 1 + x^2$ on the interval $[-1, 3]$.

!! 5. Express the limit $\lim_{n\to\infty} \left[\sum_{k=1}^{n} \sqrt{1 + \frac{2k}{n}} \cdot \ln\left(1 + \frac{2k}{n}\right) \cdot \frac{2}{n} \right]$ as a definite integral.

6. Suppose $\int_1^4 f(x)\,dx = 8$ and $\int_1^6 f(x)\,dx = 5$. Evaluate the following integrals.

(a) $\int_4^1 -3f(x)\,dx$

(c) $\int_4^6 f(x)\,dx$

(b) $\int_4^4 5f(x)\,dx$

(d) $\int_6^4 2f(x)\,dx$

7. Evaluate the following definite integral. Use the area interpretation of the integral.

$$\int_{-3}^{3} \sqrt{9-x^2}\,dx$$

2 Review 2: Fundamental theorem of calculus

Keywords: *integration, definite integral, indefinite integral, Fundamental Theorem of Calculus, antiderivative*

1. Consider the expressions $\displaystyle\int_a^b f(t)\,dt$ and $\displaystyle\int f(t)\,dt$.

   (a) What does each mean?

   (b) How are the expressions different? Give as much detail as possible.

2. State the Fundamental Theorem of Calculus. (What is a "theorem?")

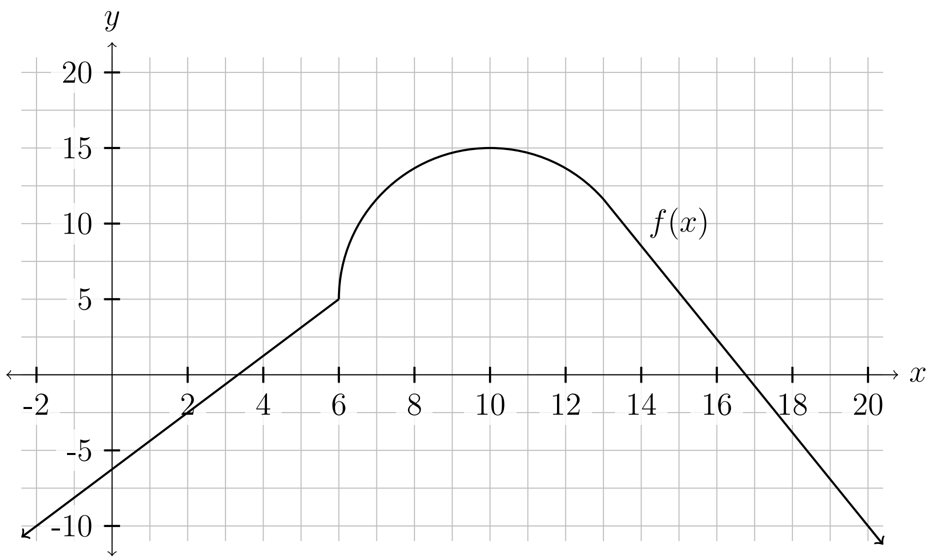
3. Do you find it harder to take an integral or a derivative? Why? Use examples to show what you mean.

4. Compute $f'(x)$, where f is defined as the function

$$f(x) = \int_1^{x^4} 2t\ln(t^2+5)\,dt.$$

   Explain which theorems, and which differentiation techniques you are using. Hint: Let $g(x) = \displaystyle\int_1^x 2t\ln(t^2+5)\,dt$ and $h(x) = x^4$. Then you will have $f(x) = g(h(x))$.

5. The graph of $f(x)$ is displayed below. Define $g(x) = \int_0^x f(t)dt$.



(a) On what intervals is g increasing?

(b) Find the local extrema of g. Indicate if each is a local minimum or local maximum.

(c) On what intervals is g concave upward? Hint: $g'' > 0$ implies $g' = f$ is increasing.

(d) Sketch a possible graph of g.

6. Evaluate the integrals below.

   (a) $\displaystyle\int_1^4 \left(3\sqrt{x} - \frac{2}{x}\right)\,dx$

   (b) $\displaystyle\int_0^\pi -2\cos(x)\,dx$

   (c) $\displaystyle\int_1^0 e^x\,dx$

   (d) $\displaystyle\int_{-3}^2 (3x^2 + 4x)\,dx$

   !! (e) $\displaystyle\int_{-1}^1 4|x^3 + \sin(\pi x)|\,dx$

3 Working with integrals

Keywords: *odd functions, even functions, symmetry, integration, average value, symmetric functions*

1. Use symmetry arguments to evaluate the following integrals.

   (a) $\displaystyle\int_{-1}^{1} 3|x^3|\,dx$

   (b) $\displaystyle\int_{-\pi}^{\pi} \sin^3(x) + \cos(x)\,dx$

   (c) $\displaystyle\int_{-2}^{2} (3x^7 + 2x^5 - 5x^4 + x^3 - x)\,dx$

   (d) $\displaystyle\int_{-5}^{5} \frac{2x^3 - x}{x^4 + 1}\,dx$

2. Find the average value of the following functions on the given interval.

   (a) $f(x) = x^4$ on $[0, 5]$.

   (b) $f(x) = \dfrac{2}{x}$ on $[1, e]$.

   (c) $f(x) = \dfrac{1}{x^2 + 1}$ on $[-\frac{1}{2}, 1]$.

   (d) $f(x) = \sin(4x)$ on $[-\frac{\pi}{2}, \frac{3\pi}{4}]$.

3. Find all points at which the given function equals its average value on the given interval.

   (a) $f(x) = 4 - x$ on $[0, 2]$.

   (b) $f(x) = \dfrac{1}{x}$ on $[1, 5]$.

   (c) $f(x) = 2 - |x|$ on $[-2, 2]$.

4. The elevation of a hiking trail is given by $f(x) = 2x^3 - 3x^2 + 10$, where x measures the horizontal distance in miles from the trail head at $x = 0$. What is the average elevation of the trail for the first three miles?

4 Substitution

Keywords: *integration, substitution, trigonometric functions, exponential functions*

1. Use the given substitutions to find the following indefinite integrals. Check your work by differentiating.

   (a) $\displaystyle\int (6x+1)\sqrt[3]{3x^2+x}\,dx$ with $u = 3x^2 + x$.

   (b) $\displaystyle\int \sin^3(x)\cos(x)\,dx$ with $u = \sin(x)$.

   (c) $\displaystyle\int \tan^2(\theta)\sec^2(\theta)\,d\theta$ with $u = \tan(\theta)$.

2. Use integration by substitution to evaluate the following integrals.

   (a) $\displaystyle\int xe^{x^2}\,dx$

   (b) $\displaystyle\int x\sqrt{x^2+4}\;dx$

   (c) $\displaystyle\int \frac{f'(x)}{f(x)}\;dx$

!! (d) $\displaystyle\int x\sqrt{4-x}\,dx$

3. Consider the integral $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$.

   (a) What is a good choice of u when doing substitution?

   (b) What are the new limits of integration in terms of u?

   (c) Evaluate the integral.

4. Evaluate the following integrals. Don't forget that when the variable of integration is changed from x to u, the limits of integration must also be expressed in terms of u.

   (a) $\displaystyle\int_e^{e^2} \frac{\ln^2(x)}{x}\,dx$

   (b) $\displaystyle\int_0^1 x^2\sqrt{2x^3+1}\,dx$

   (c) $\displaystyle\int_0^{\pi/16} \cos^2(8\theta)\sin(8\theta)\,d\theta$

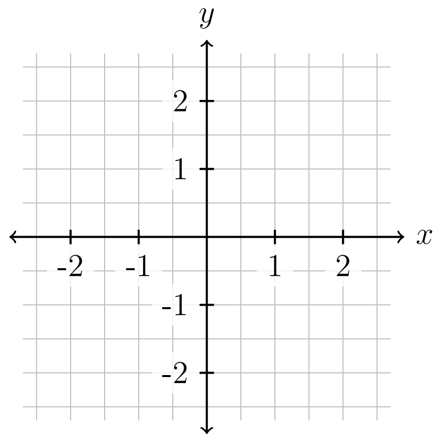
5. Write down and solve a definite integral which can be nicely solved using substitution. How elegant can you make it? Can you come up with one that requires three substitutions to solve?

5 Regions between curves and volumes

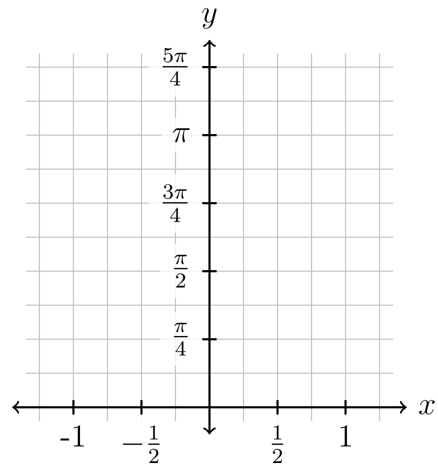
Keywords: *area, region between curves, area of compound regions, volume, slicing method*

1. Sketch the given curves and indicate the region that is bounded by both. Then set up and evaluate an integral representing the area of the region.

   (a) $y = x$ and $y = x^2 - 2$.

   (b) $x = \sin(y)$ and $x = \cos(y)$ for $\frac{\pi}{4} \le y \le \frac{5\pi}{4}$.

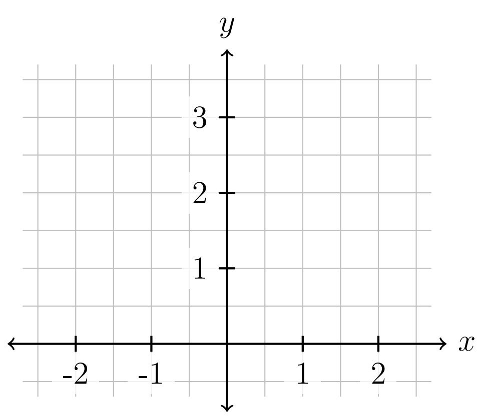
   

2. Consider region bounded by the curves $y = 2^x$, $y = 3 - x$, and the y-axis.

   (a) Draw the curves and shade in this region.

   (b) Find the area of this shaded region by integrating with respect to x.

   (c) Find the area of the region by integrating with respect to y. You may assume that $x\ln(x) - x$ is an antiderivative of $\ln(x)$.

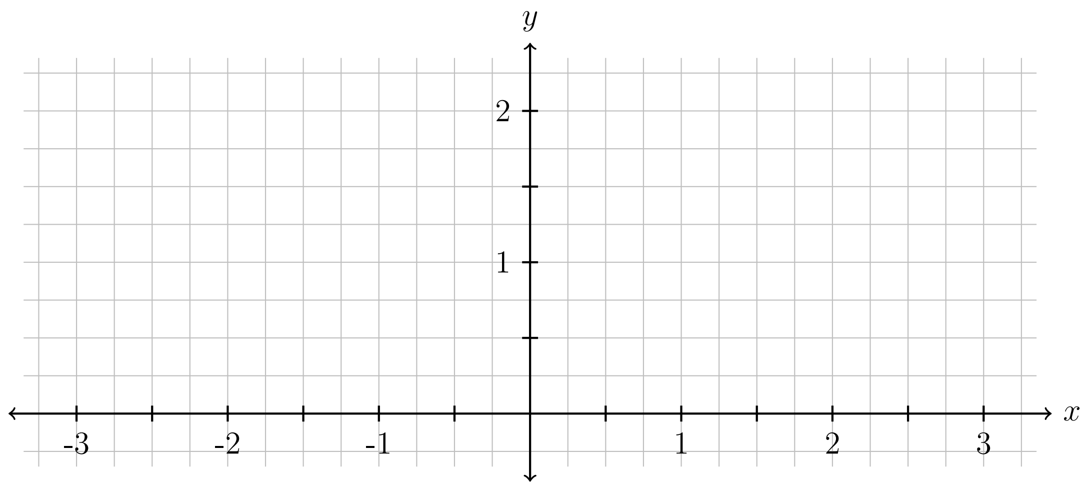
3. When setting up the integral for the area of a region bounded by curves, how do you determine if you should integrate with respect to x or with respect to y? In what cases would you / would you not draw a picture? Why?

4. (a) Give a pair of functions that intersect at least twice, so that the area of the region they both bound is most easily found by integrating with respect to x. Draw their graphs below.

   (b) Give a different pair of functions (don't just change the variables from part (a)) that intersect at least twice, so that the area of the region they both bound is most easily found by integrating with respect to y. Draw their graphs below.

5. Let R be the region bounded by the circle of radius 1 centered at $(2, 1)$.

   (a) Draw the circle and shade in the region R below.



   (b) Sketch the solid S obtained by revolving R about the y-axis.

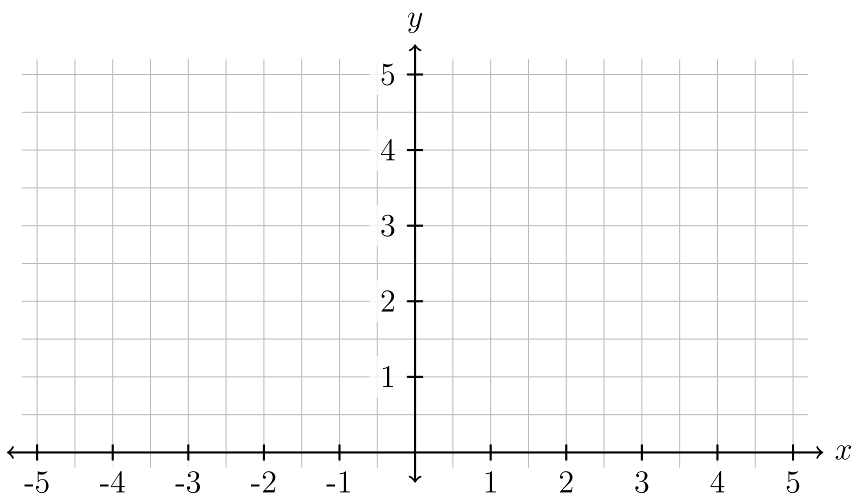
   (c) Set up the integral representing the volume of the solid S.

6 Volumes by slicing

Keywords: *area, region between curves, volume, slicing method, solid of revolution*

1. Let R be the region bounded by the curves $y = \sqrt{25 - x^2}$ and $y = 0$.

   (a) Draw the curves and shade in the region R below.



   (b) Suppose R is revolved around the x-axis to create a solid shape.

      i. What does the cross-section of this shape look like at $x = 3$?

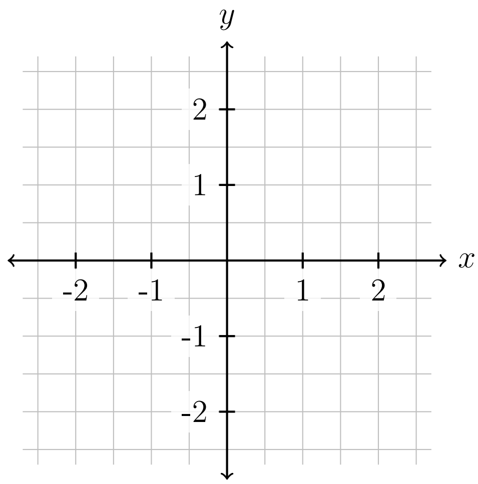
      ii. Set up the integral for the volume of this solid.

   (c) Verify that your answer agrees with the volume formula of a sphere.

2. Let R be the region bounded by the parabolas $x = 1 - y^2$ and $x = y^2 - 1$.

   (a) Draw the parabolas and shade in the region R below.

   (b) Set up the integral representing the volume of the solid obtained by rotating the region R about the line $x = -2$. It may help to draw the shape of a cross-section of the solid.

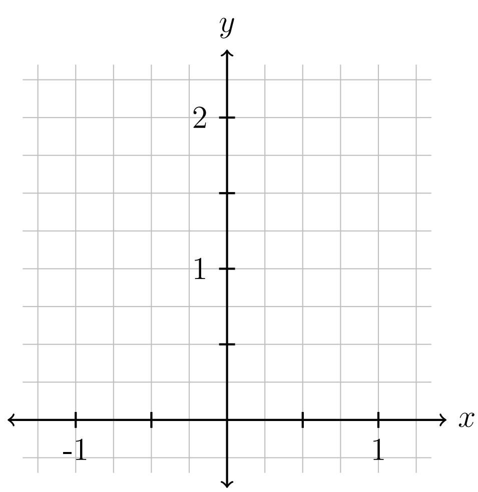


3. Let S be the solid whose base is bounded by the curves $y = x^2$ and $y = 2 - x^2$, and whose cross-sections perpendicular to the base are squares.

   (a) Draw the curves and shade in the base below.

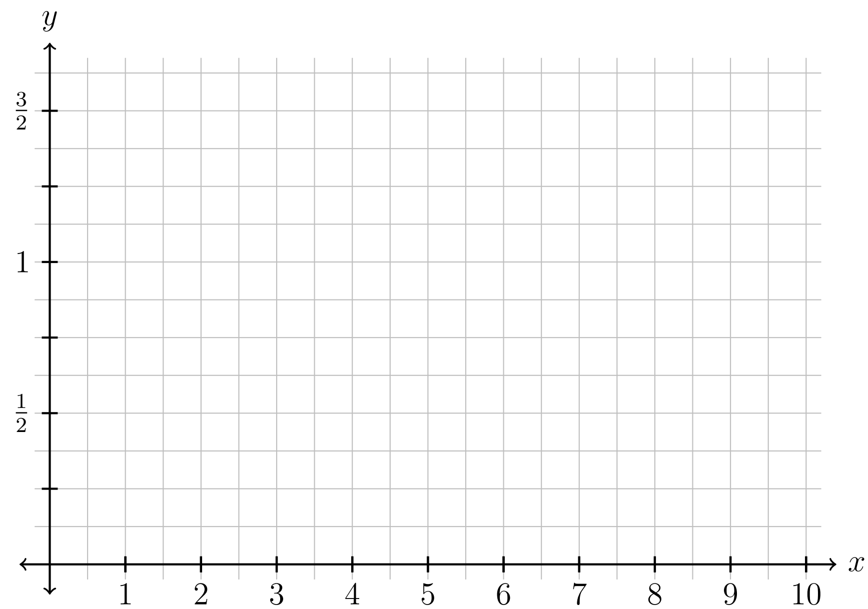
   (b) Make a sketch of what you think the solid S looks like.

   (c) Set up the integral representing the volume of S.



4. Let R be the region bounded by $y = 1/x$, $y = 0$, $x = e$ and $x = e^2$.

   (a) Make a sketch of the region.



   (b) Set up the integral for the volume of the solid obtained by rotating the region R around $y = -1$.

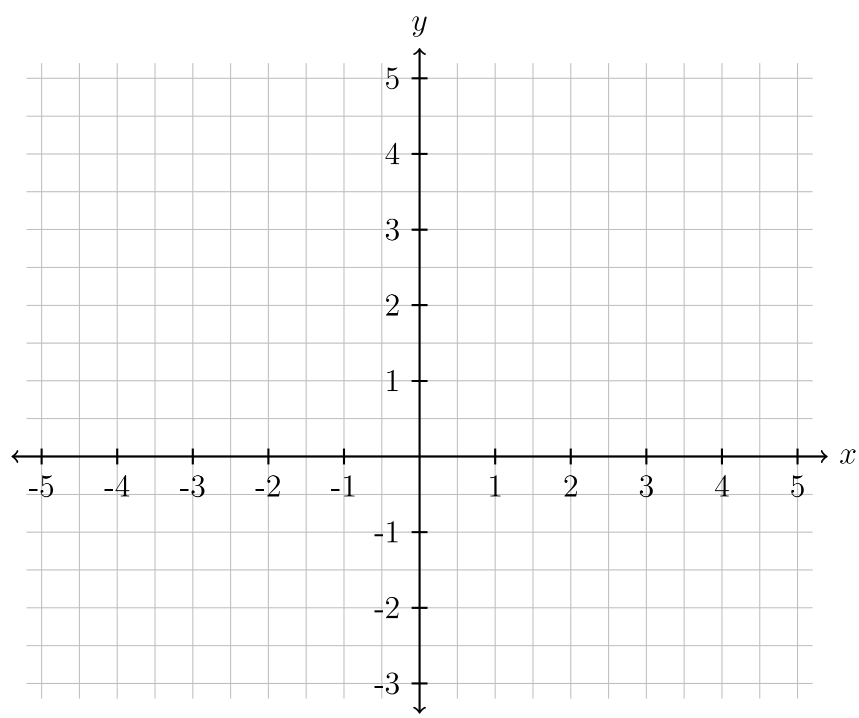
   (c) Set up the integral for the volume of the solid obtained by rotating the region R around $y = 1$.

7 Length of curves and work

Keywords: *arc length, length of curves, physical applications, work, pumps, springs*

1. (a) Find the arc length of the curve $y = -\frac{3}{4}x + 1$ where x is in $[-4, 4]$.

   (b) Sketch the curve below.



   (c) Check whether your answer from (a) is correct by using the Pythagorean theorem on (b).

2. For each of the following curves, set up an integral representing its arc length.

   (a) $y = \dfrac{2}{3}(x-1)^2$ on the interval $[1, 4]$.

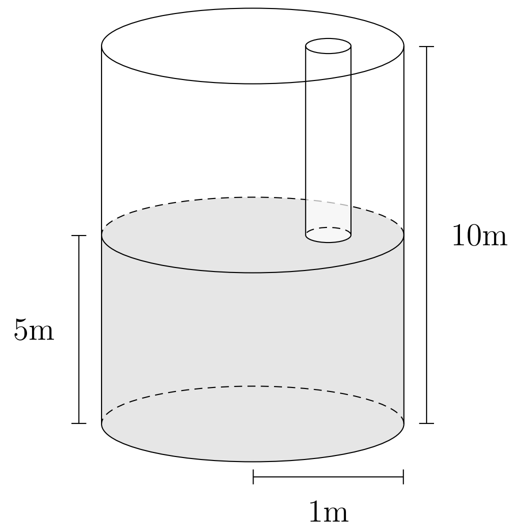
   (b) $y = x^{5/4}$ on the interval $[0, 1]$.

   (c) $y = 1/x$ on the interval $(0, 4]$. Knowing the graph of $1/x$, what is the arc length?

3. Compute the arc length of the curve $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$ on the interval $[1, 3]$.

4. The hyperbolic cosine function $\cosh(x)$ is defined as $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$. Find the arc length of $\cosh(x)$ on the interval $[-\ln(2), \ln(2)]$.

5. A tank shaped like a cylinder, with a height of 10 meters and a radius of 1 meter, is half filled with water. A hose is dropped in from the top to the surface of the water, where it floats, as in the diagram below.



(a) Set up and evaluate an integral representing the work needed to pump the water out of the tank through the hose. The density of water is 1000 kg/m^3, and you may assume that the acceleration due to gravity is 10 m/s^2.

(b) Repeat part (a) in the situation that only one fifth of the tank has been filled.

(c) Repeat part (a) in the situation that the tank is a rectangular prism, whose base is a rectangle of width 2 m and length 3 m (the height is still 10 m). Sketch the diagram first.

6. A spring has a natural length of 0.2 m. A force of 10 N is required to stretch the spring to a length of 0.3 m.

   (a) If we apply a force of 27 N to stretch the spring, what is the spring's length? Hint: Recall that Hooke's law states the force required to keep a spring stretched or compressed x units from its natural length is $F(x) = kx$, where k is the spring's constant.

   (b) How much work is done to stretch the spring from 0.2 m to 0.3 m?

!! 7. Four Brothers, the new fast food chain, was making a delivery of their house root beer. The root beer initially filled a container that is in the shape of an inverted cone with a height of 10 m and a top radius of 2 m. The crew of Four Brothers is using a Super Suction machine to pump the root beer to the top of the container. The motor on the pump stopped working when the height of the remaining root beer in the container was 3 m. Using the facts that the density of the house root beer is 160 kg/m^3 and the acceleration due to gravity on earth is approximately 10 m/s^2, set up an integral that describes the amount of work that was done by the pump.

8 Integration by parts

Keywords: *integration by parts, definite integral, indefinite integral*

1. Use integration by parts twice to integrate $\int x^2 \sin(2x)\, dx$.

2. For each of the following integrals, how many iterations of integration by parts is required to evaluate it? Do not evaluate the integrals, but explain your reasoning.

   (a) $\int x^5 \sin(x)\, dx$

   (b) $\int (y^5 + 3y^3 - y^2) \cos(3y)\, dy$

3. Evaluate the integral $\displaystyle\int \frac{x}{\sqrt{x-2}}\,dx$

   (a) using substitution.

   (b) using integration by parts.

   (c) Reconcile the results of (a) and (b).

4. Use integration by parts to evaluate the following integrals.

   (a) $\displaystyle\int (x+3)\sin(2x)\,dx$

   (b) $\displaystyle\int (2x-1)e^x\,dx$

   (c) $\displaystyle\int_1^e \ln(2x)\,dx$

5. Use integration by parts twice to integrate $\displaystyle\int e^{3x}\cos(3x)\;dx$.

9 More integration by parts and trigonometric integrals

Keywords: *integration by parts, trigonometric integration, trigonometric functions*

1. Use integration by parts to evaluate the following integrals.

   (a) $\displaystyle\int \sin(2z)\cos(3z)\ dz$

   (b) $\displaystyle\int_0^{\frac{1}{2}} \arccos(\theta)\, d\theta$

2. (a) State the half angle identities used to integrate $\sin^2(x)$ and $\cos^2(x)$.

(b) State the Pythagorean identities.

(c) Use the above identities to evaluate the following integrals.

i. $\displaystyle\int 4\cos^4(x)\,dx$

ii. $\displaystyle\int \sin^2(\phi)\cos^2(\phi)\,d\phi$

iii. $\displaystyle\int \tan^2(\theta)\,d\theta$

3. Use trigonometric identities and substitution to evaluate the following integrals.

   (a) $\displaystyle\int \sin^3(x)\cos^2(x)\,dx$

   (b) $\displaystyle\int 6\sin^2(y)\cos^5(y)\,dy$

4. Evaluate $\displaystyle\int \sec^4(\theta)\,d\theta$. Hint: Use the Pythagorean identity $\sec^2(x) = 1 + \tan^2(x)$.

10 Partial fractions and improper integrals

Keywords: *partial fractions, factoring, long division, polynomials, improper integral, infinite limit of integration, infinite intervals, unbounded integrands*

1. Write out the form of the partial fraction expansion for the following functions. Do not determine the numerical values of the coefficients.

   (a) $\dfrac{x^2 - 2x + 3}{x^3 - x^2 - 6x} =$

   (b) $\dfrac{x+1}{x^3 + 2x^2} =$

   (c) $\dfrac{5}{(x^2 - 1)(x^2 + 1)} =$

2. Evaluate the following integrals. You will have to use partial fractions, factoring, and polynomial division.

   (a) $\displaystyle\int \frac{1}{(x-3)(x+1)^2}\,dx$

   (b) $\displaystyle\int \frac{dx}{x^2 - 7x + 10}$

3. Rewrite each the following improper integrals as a limit. If it converges, evaluate it. The first one is done for you.

   (a) $\displaystyle\int_1^\infty \frac{dx}{x(x+1)} = \lim_{b\to\infty} \int_1^b \frac{dx}{x(x+1)} = \lim_{b\to\infty} \int_1^b \frac{1}{x} - \frac{1}{x+1}\,dx = \lim_{b\to\infty} \ln|x| - \ln|x+1|\Big|_1^b = \lim_{b\to\infty} \ln\left|\frac{x}{x+1}\right|\Bigg|_1^b = \lim_{b\to\infty} \ln\left|\frac{b}{b+1}\right| - \ln\left|\frac{1}{2}\right| = \ln(1) + \ln(2) = \ln(2)$ since the partial fraction decomposition of $\dfrac{dx}{x(x+1)}$ is $\dfrac{1}{x} - \dfrac{1}{x+1}$.

   (b) $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x}}$

   (c) $\displaystyle\int_2^3 \frac{dy}{y-3}$

   (d) $\displaystyle\int_1^\infty \frac{\ln(x)}{x^3}\,dx$

   (e) $\displaystyle\int_{-\infty}^0 x^2 e^{x^3}\,dx$

4. For which values of p does the integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge and for what values of p does it diverge? Hint: The antiderivative of x^{-p} takes one form when $p \neq 1$ and another form when $p = 1$.

5. Using comparison, determine if the following integrals converge or diverge.

   (a) $\displaystyle\int_1^\infty \frac{\cos^2(x)}{x^2}\,dx$

   (b) $\displaystyle\int_1^\infty \frac{dx}{\ln(x)}$

6. Gabriel's horn, also called Torricelli's trumpet, is the surface of revolution of the function $y = 1/x$ about the x-axis for $x \geq 1$.

   (a) Find the volume of Gabriel's horn.

   (b) The surface area of a surface of revolution defined by the function f, revolved around the x-axis from $x = a$ to $x = b$ is given by the formula

   $$\int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx.$$

   Find the surface area of Gabriel's horn or show that it is infinite. Hint: Use comparison.

   (c) Reconcile the results of (a) and (b).

11 Introduction to sequences

Keywords: *sequences, recursive sequences, limits of sequences, convergence, divergence*

1. Find an explicit formula for the nth term of the following sequences. The first term given in each sequence is for $n = 1$.

   (a) $16, 25, 36, 49, \ldots$

   (b) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

   (c) $1, \frac{-1}{4}, \frac{1}{27}, \frac{-1}{256}, \ldots$

   (d) $-1, 1, -1, 1, -1, 1, \ldots$

   (e) $0, 1, 0, 1, 0, \ldots$

   (f) $1, 1, 2, 2, 3, 3, 4, \ldots$

2. Find the limit of each sequence as n goes to infinity or state that it does not exist.

   (a) $a_n = \dfrac{n^3}{n^4+1}$

   (b) $b_n = \dfrac{\ln(n)}{n}$

   (c) $c_n = \dfrac{\ln(\frac{1}{n})}{n}$

   (d) $d_n = \dfrac{(-1)^n}{\sqrt{n}}$

   !! (e) $e_n = \left(1 + \dfrac{4}{n}\right)^n$

   (f) $f_n = \dfrac{\sin(\frac{n\pi}{8})}{\sqrt{n}}$

3. Let a_n be given by the recurrence relation[1] $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, $a_0 = 1$.

   (a) Calculate the first three terms of the sequence.

   (b) Assuming the limit exists, calculate $\lim_{n\to\infty} a_n$.

!! 4. Let a_n be the sequence given by $a_n = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\ldots\left(1 - \frac{1}{n}\right)$.

   (a) Calculate the first four terms of the sequence.

   (b) Find the limit of the sequence as n goes to infinity.

[1]This is perhaps the first algorithm used for approximating $\sqrt{S}$, known as the "Babylonian method". It is a quadratically convergent algorithm, which means that the number of correct digits of the approximation roughly doubles with each iteration. It proceeds as follows: Start with an arbitrary positive start value x_0 (the closer to the root, the better). Let x_{n+1} be the average of x_n and S/x_n. Repeat until the desired accuracy is achieved.

5. Give an example of each of the following sequences. Use a different one for each!

(a) non-increasing sequence

(b) increasing sequence

(c) non-decreasing sequence

(d) decreasing sequence

(e) constant sequence

(f) monotonic sequence

(g) sequence that is bounded below

(h) sequence that is bounded above

(i) bounded sequence

(j) convergent sequence

6. Notice that $0.9 = \frac{9}{10}$, $0.99 = \frac{9}{10} + \frac{9}{100}$ and so on.

(a) Use this pattern to define a sequence $\{a_n\}$ such that $\displaystyle\sum_{n=1}^{\infty} a_n = 0.99999\ldots$.

(b) Use this pattern to define a sequence $\{a_n\}$ such that $\displaystyle\sum_{n=1}^{\infty} a_n = 0.1234123412\ldots$.

12 Introduction to series

Keywords: *series, convergence, divergence, partial sums, infinite series, geometric series, telescoping series*

1. Consider the sequence

$$4,\ -\frac{4}{5},\ \frac{4}{25},\ -\frac{4}{125},\ \ldots$$

   (a) Find a formula for the general term a_n of this sequence.

   (b) Find $\lim\limits_{n\to\infty} a_n$.

   (c) Does the series $\sum\limits_{n=1}^{\infty} a_n$ converge or diverge? If it converges, find its sum.

2. Determine whether the following telescoping series converge or diverge by finding a formula for the k-th partial sum and evaluating its limit if the limit exists.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

3. Determine whether the following geometric series converge or diverge. For each series find the ratio r and its first term a. If the series converges, compute its sum.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^n}{9^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^{2n}}{3^n}$

4. Evaluate the following sums (do not use brute force).

   (a) $\displaystyle\sum_{k=0}^{12} 2^k$

   (b) $\displaystyle\sum_{n=1}^{10} \left(\frac{4}{7}\right)^n$

   (c) What happens if we take these two sums up until infinity? Do they converge or diverge?

5. Write the repeating decimal $0.\overline{456}$ first as a geometric series and then as a simplified fraction.

6. **Koch Snowflake Fractal** The fractal called the Koch Snowflake is constructed as follows: Let I_0 be an equilateral triangle with sides of length 1. The figure I_1 is obtained by replacing the middle third of each side of I_0 with a new outward equilateral triangle with sides of length 1/3. The process is repeated where I_{n+1} is obtained by replacing the middle third of each side of I_n with a new equilateral triangle with sides of length $1/3^{n+1}$. The limiting figure as $n \to \infty$ is called the Koch Snowflake.

   (a) First draw the first three stages of the construction.

   (b) Let L_n be the perimeter of I_n. Show that $\lim\limits_{n\to\infty} L_n = \infty$, i.e. the perimeter is infinite.

   (c) Let A_n be the area of I_n. Find $\lim\limits_{n\to\infty} A_n$. Show that it exists and is finite!

   (d) Compare the Koch Snowflake with Gabriel's horn.

13 The divergence, p-series, and ratio tests

Keywords: *series, divergence test, p-test, ratio-test, harmonic series, geometric series*

1. If $\lim_{k\to\infty} a_k = 1$, what can you say about $\sum_{k=1}^{\infty} a_k$?

2. If the terms of a positive series decrease to zero, then does the series converges? Given an example or counter example to support your argument.

3. For what values of p does the series $\sum_{k=10}^{\infty} \frac{1}{k^p}$ converge or diverge?

4. Determine whether the following statements are true or false.

   (a) The series $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n = 5$.

   (b) If the sequence $\{a_n\}$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.

   (c) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges.

   (d) The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ converges.

5. Use the Ratio test to determine if the following series converge.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{n!}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{5^n}$

6. Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{5}{k}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^3 - 1}{k^3 + 1}$

(c) $\displaystyle\sum_{k=2}^{\infty} \frac{k^e}{k^\pi}$

(d) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{\pi}{e}\right)^k$

(e) $\displaystyle\sum_{k=1}^{\infty} \left(e^{1/k} - e^{1/(k+1)}\right)$

(f) $\displaystyle\sum_{k=1}^{\infty} \frac{3}{(k^2)!}$

(g) $\displaystyle\sum_{k=1}^{\infty} \frac{5}{\sqrt{k}}$

(h) $\displaystyle\sum_{k=1}^{\infty} \frac{2^{k+1} + (-2)^k}{5^k}$

14 The comparison test

Keywords: *series, comparison test, limit comparison test*

1. Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ converges.

   (a) If $a_n < b_n$ for all n, what can you say about $\sum a_n$? Why?

   (b) If $a_n > b_n$ for all n, can you say anything about $\sum a_n$? Why?

   (c) What could you say in cases (a) and (b) if the series had negative terms instead?

2. Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{-1}{n^3 - 7n^2 + 17n + 20}$ converges or diverges.

3. Consider the series $\sum_{n=2}^{\infty} \frac{n+1}{n^2}$.

   (a) Use the direct comparison test to determine if it converges or diverges.

   (b) Now use the limit comparison test and see if you arrive at the same conclusion.

4. Consider the series $\sum_{n=2}^{\infty} \frac{n+1}{n^3}$.

   (a) Use the limit comparison test to determine if it converges or diverge.

   (b) Why can't you use the direct comparison test here?

5. Determine if the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^3}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{7k^4 + 3k^2 + 1}{11k^5 - 4k^3 + 9k^2 - 1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{7^n - 5^n}$

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - 3k}$

15 The alternating series test and approximating functions with polynomials

Keywords: *series, alternating series test, absolute convergence, conditional convergence, applications, approximation, power series, Taylor polynomials, linear approximation, quadratic approximation*

1. Determine whether the following series converge or diverge.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

   (b) $\displaystyle\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right|$

   (c) Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converge absolutely, converge conditionally, or diverge?

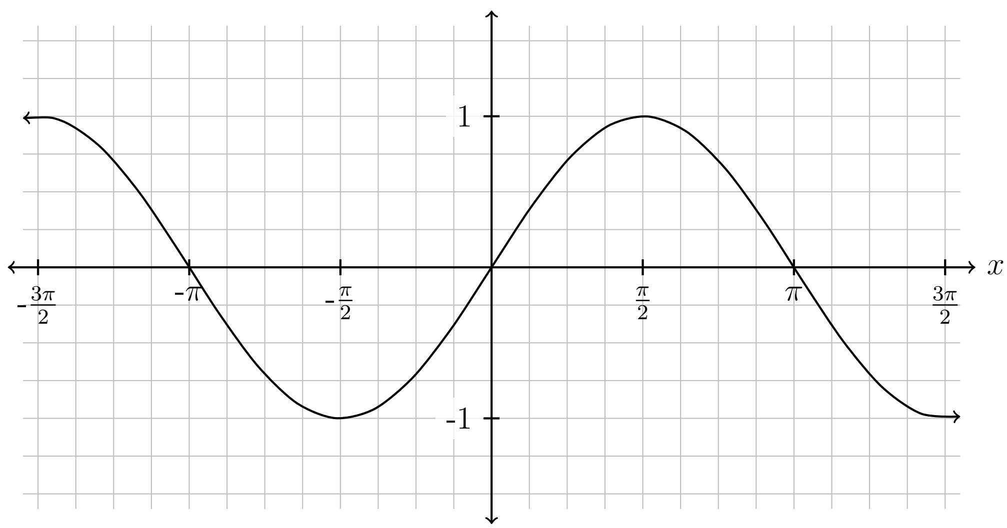
2. Determine whether the given series converges conditionally, converges absolutely, or diverges. Justify your response by specifying which convergence tests you use and verifying that the conditions of each used test are satisfied.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$

3. (a) Find the n-th order Taylor polynomial of the function $f(x) = \sin(x)$ centered at $a = 0$ for $n = 1, 2,$ and 3.

   (b) Graph the above Taylor polynomials in the coordinate system below.



4. (a) Find the 3rd-degree Taylor polynomial of $f(x) = \sqrt{x}$ centered at $a = 1$.

   (b) Use your result from part (a) to approximate $\sqrt{0.9}$. You do not need to simplify your answer.

5. Match the functions with their second degree Taylor polynomials centered at zero. Give reasons for your choices!

$a(x) = \sqrt{1+2x}$	$p_2(x) = 1 + 2x + 2x^2$
$b(x) = \dfrac{1}{\sqrt{1+2x}}$	$q_2(x) = 1 - 6x + 24x^2$
$c(x) = e^{2x}$	$r_2(x) = 1 + x - \dfrac{x^2}{2}$
$d(x) = \dfrac{1}{1+2x}$	$s_2(x) = 1 - 2x + 4x^2$
$e(x) = \dfrac{1}{(1+2x)^3}$	$t_2(x) = 1 - x + \dfrac{3}{2}x^2$
$f(x) = e^{-2x}$	$u_2(x) = 1 - 2x + 2x^2$

6. (a) Find the 1st, 2nd, and 3rd degree Taylor polynomials of $f(x) = x^2$ centered at $a = 0$.

   (b) Find the 2nd degree Taylor Polynomial of $f(x) = x^2$ centered at $a = 1$.

   (c) Compare your results from (a) and (b). What can you conclude?

16 Power series

Keywords: *power series, radius of convergence, interval of convergence, combining power series, differentiating power series, integrating power series*

1. Determine the radius of convergence of the following power series. Then test the endpoints to determine the interval of convergence.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{(x-1)^k}{k}$

   Check endpoints:

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-1)^k}{k3^k}$

   Check endpoints:

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n\sqrt{n}}$

Check endpoints:

!! 2. Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^n}$.

3. Use the geometric series $\dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k$ for $|x| < 1$, to find a power series representation for the following functions.

   (a) $f(x) = \dfrac{1}{1+x}$

   (b) $g(x) = \dfrac{1}{1-x^2}$

   (c) $h(x) = \dfrac{x^3}{1+x}$

   (d) $k(x) = \displaystyle\int \frac{x^3}{1+x} dx$

   (e) What are the radii of convergence of the power series above?

!! 4. A tortoise and a hare wanted to determine which was the better animal so the tortoise proposed a 100 m race. The hare thought the tortoise was foolish for wanting to race and told the tortoise that he should just forfeit. The tortoise said, "if you are so confident in your victory, you should give me a head start." The hare replied, "you can have a 20 m head start. It won't matter since I'm much faster than you." The tortoise then said, "by giving me a head start you have lost the race." The confused hare asked, "how could that be? I'm significantly faster than you." The tortoise explained, "no matter how fast you are, it will take you some amount of time to reach the 20 m mark and in that time I would have moved forward some distance. It will take you some more time to go that distance in which I would have moved forward some more. In the time it takes you to reach where I am then, I will have moved closer to the finish. We can repeat this over and over. Every time you reach the position I was just at, I will have moved some distance forward and will still be ahead. You will never be able to pass me even if we do this an infinite amount of times since each time I will be some distance ahead of you." The hare, believing he would be defeated, decided to forfeit giving the hare the victory. Explain to the hare, using sequences and series, why even though what the tortoise said was true, the hare would have still won the race.

5. Let $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}$.

   (a) Show that $f(x)$ has infinite radius of convergence.

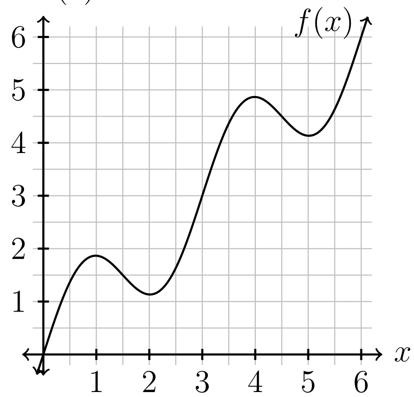
   (b) Show that $f'(x) = xf(x)$.

17 Numerical integration

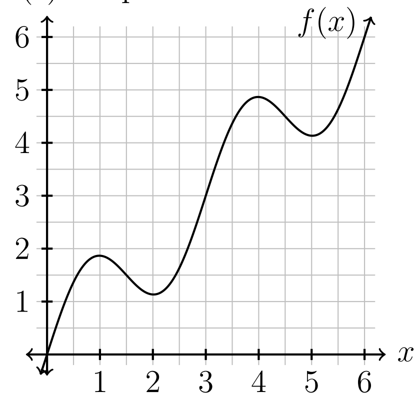
Keywords: *numerical integration, midpoint rule, trapezoid rule, Simpson's rule*

1. Complete the figures by graphing the missing approximations of the indicated numerical integration techniques using $n = 6$ subintervals.

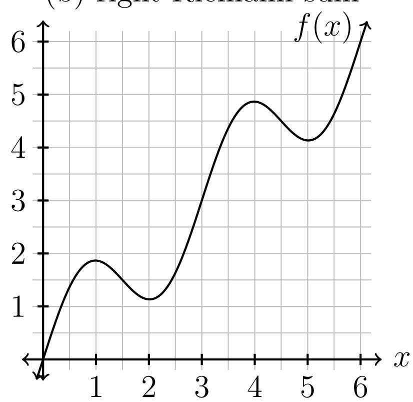
(a) left Riemann sum



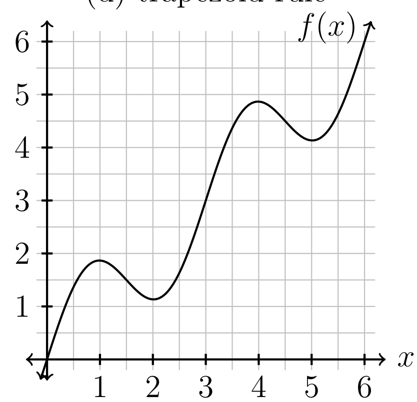
(c) midpoint Riemann sum



(b) right Riemann sum



(d) trapezoid rule



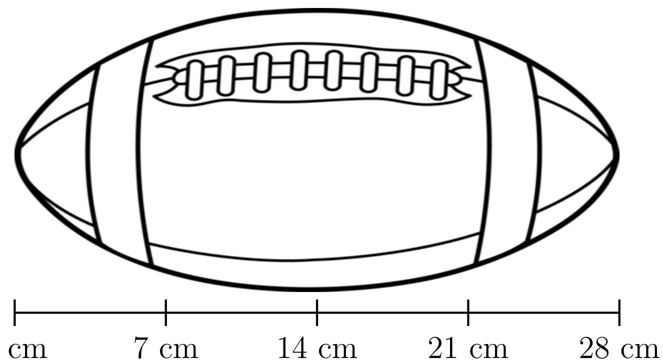
2. In the previous problem, visually which integration technique do you think gives the best approximation to $f(x)$ from $x = 0$ to $x = 6$?

3. A submarine is traveling under the polar ice cap directly towards the North Pole. The velocity v (in miles per hour) of the submarine is recorded below. Use Simpson's Rule to estimate the total distance the submarine traveled during the 8 hour period.

t	0	2	4	6	8
v	10	15	12	10	16

4. Suppose you're asked to estimate the volume of a football. You measure and find that a football is 28 cm long. You use a piece of string and measure the circumference at its widest point to be 53 cm. The circumference 7 cm from the end is 45 cm.

   (a) Use your measurements to complete the table which will assist you in estimating the volume of the football.



circumference					
radius					
cross-sectional area					

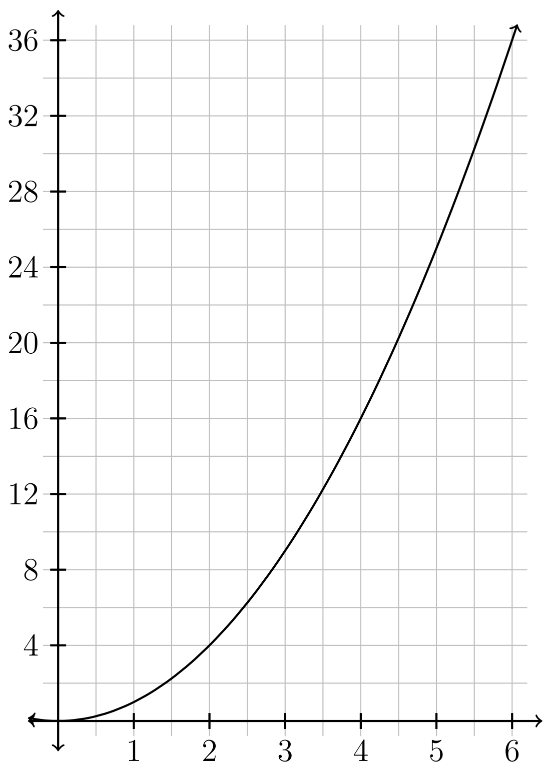
   (b) Use Simpson's rule to make your estimate.

   (c) Use the trapezoidal rule to make your estimate.

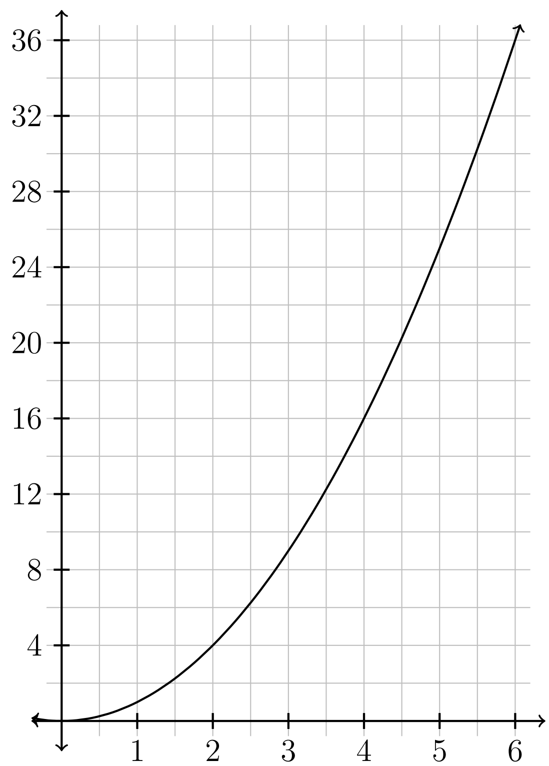
5. Using the trapezoid sum, calculate the area underneath the curve $y = x^2$ from 0 to 1, when it is split up into n intervals, then take the limit as $n \to \infty$.

6. Find the midpoint rule approximation for $\int_1^5 x^2 dx$ using n subintervals.

(a) $n = 4$



(b) $n = 8$



!! (c) Are these midpoint approximations over estimates or under estimates? Why? Calculate the error of each estimate.

18 Taylor series

Keywords: *Taylor series, series expansion, series approximation, Maclaurin series*

1. Let $f(x) = \cos(\pi x)$.

   (a) Evaluate $f(1), f'(1), f''(1), f'''(1), f^{(4)}(1)$ and $f^{(5)}(1)$.

   (b) Find the Taylor series expansion at $x = 1$ of the function $f(x)$.

2. (a) Use the Maclaurin series $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$ to find the Maclaurin series of $e^{(x^3)}$.

   (b) Use the first three non-zero terms of the series you found in the previous problem to approximate the definite integral $\displaystyle\int_0^1 e^{(x^3)}\, dx$.

3. Suppose $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k^2} = x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \cdots$

   (a) Find the first three non-zero terms of the power series representing the function $xf(x^2)$.

   (b) Use your results in part (a) to approximate the indefinite integral $\int xf(x^2)\,dx$.

!! 4. Using the Taylor series for xe^x centered at $x = 0$, show that $\sum_{n=0}^{\infty} \frac{n+1}{n!} = 2e$.

5. Consider the series $f(x) = \sum_{k=0}^{\infty} \frac{(2x+3)^k}{k!}$.

   (a) Find the derivative $f'(x)$ of the series and rewrite it in terms of $f(x)$.

   (b) Using part (a), give the nth derivative of $f(x)$. Do not simply keep taking derivatives of the series.

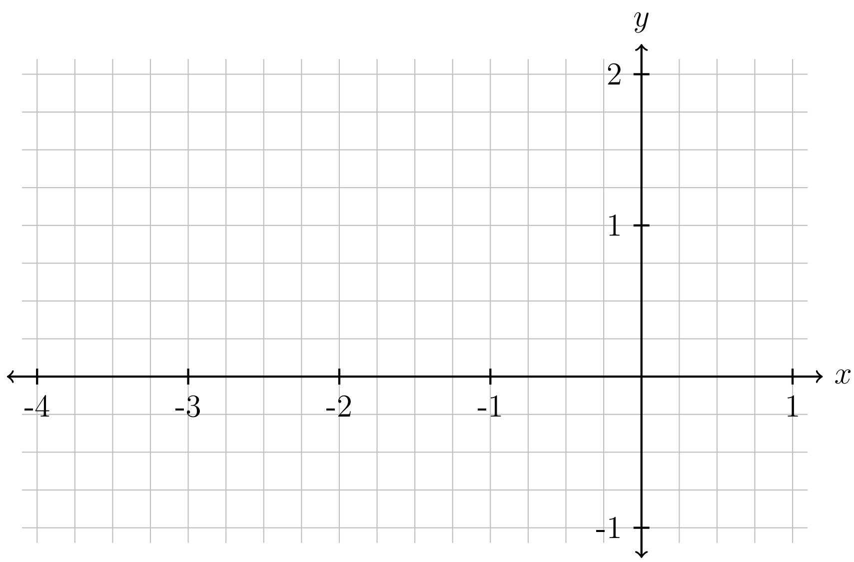
   (c) What common function is $f(x)$ equal to?

19 Parametric equations

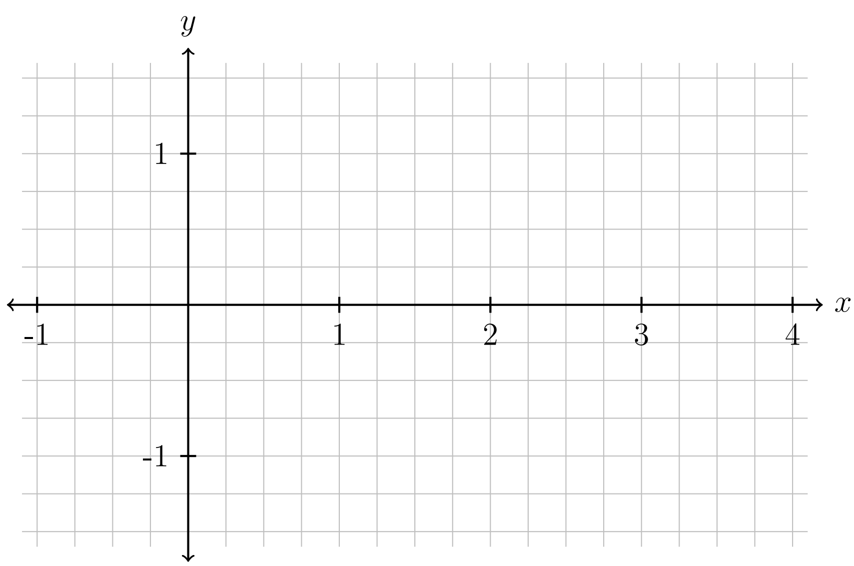
Keywords: *parametric equations, circles, arcs, ellipses, circular motion, slope*

1. Eliminate the parameter t to obtain a single equation for the parametric curves in terms of only x and y. Make a rough sketch of these curves, and indicate with arrows the direction of motion.

   (a) $x = 6t - 2$, $y = 3t$



   (b) $x = \sin(t) + 2$, $y = \cos(t)$



2. A Ferris wheel has a radius 20 m and completes a revolution in the clockwise direction at constant speed in 3 min. Assume that x and y measure the horizontal and vertical positions of a seat on the Ferris wheel relative to the coordinate system whose origin is at the low point of the wheel. Assume that the seat begins moving at the origin.

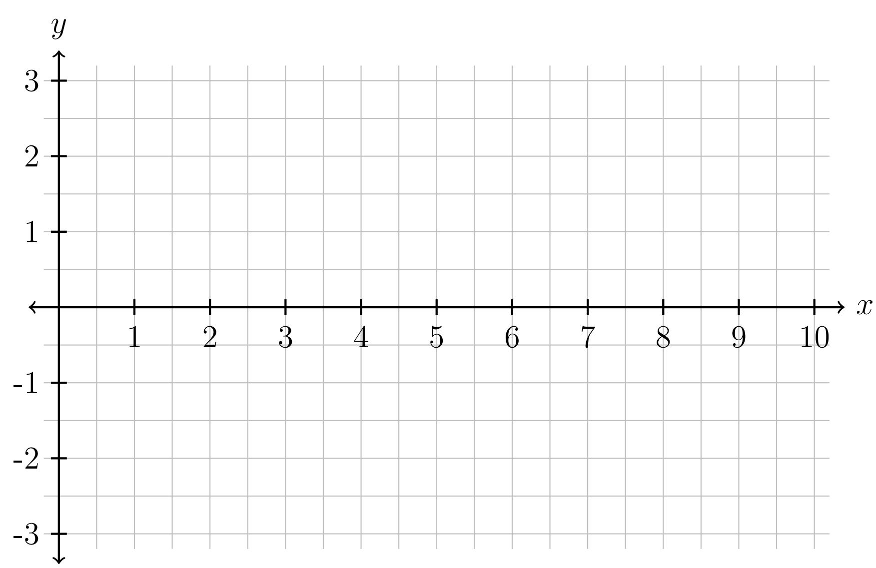
   (a) Find the parametric equations that describe the position of the seat at time t.

   (b) What is the position of the rider when $t = 1$ min?

3. Consider the parametric curve given by $x = 4t^2 + 1$ and $y = 2t$.

   (a) Determine dy/dx in terms of t and evaluate it at $t = -1$.

   (b) Make a sketch of the curve showing the tangent line at the point corresponding to $t = -1$.



4. Match each of the equations (a)-(d) with one of graphs (i)-(iv). Explain your reasoning.

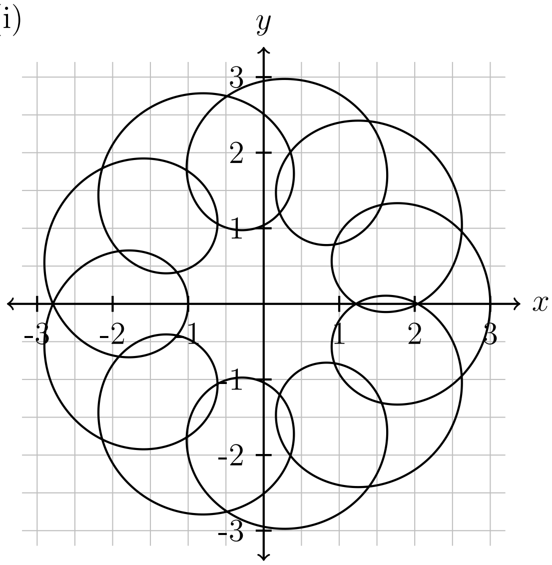
   (a) $x = t^2 - 2,\ y = t^3 - t$

   (b) $x = \cos(t + \sin(50t)),\ y = \sin(t + \cos(50t))$

   (c) $x = t + \cos(3t),\ y = t - \sin(3t)$

   (d) $x = 2\cos(t) + \cos(10t),\ y = 2\sin t + \sin(10t)$

(i)



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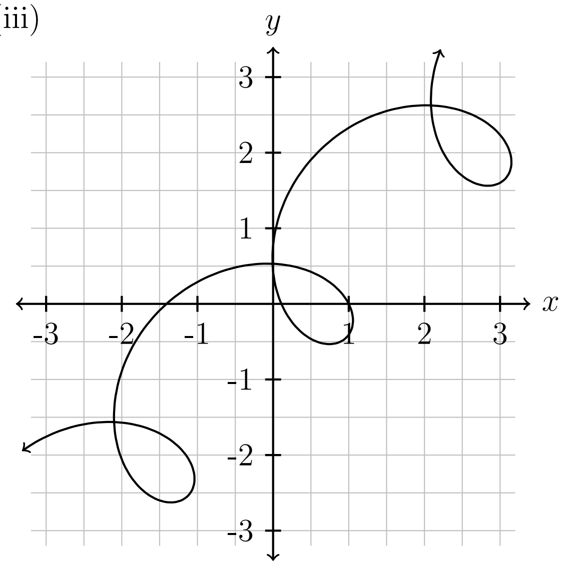
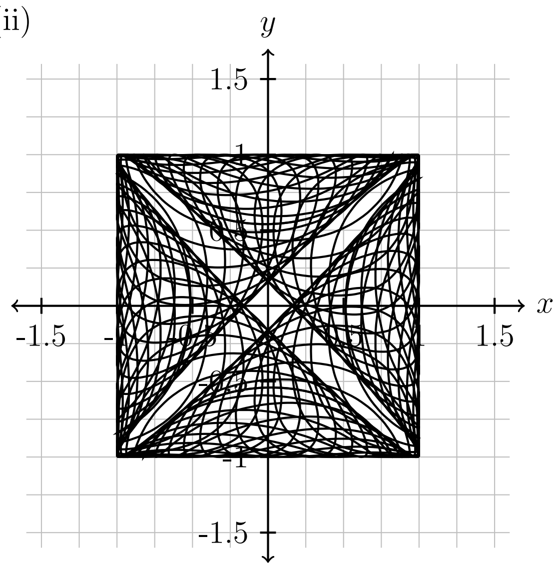


(ii)



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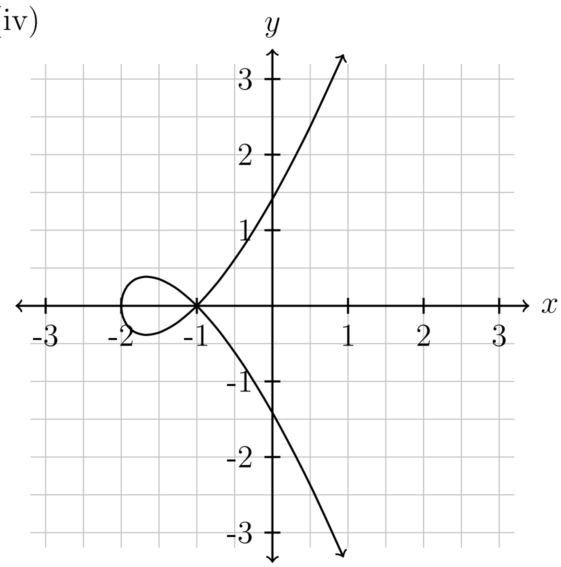


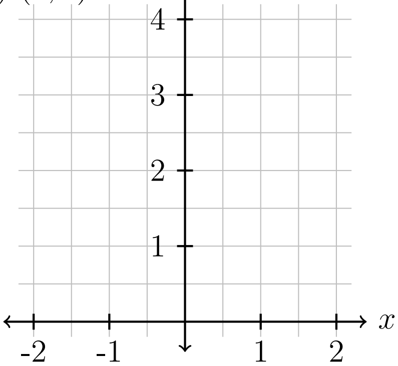
20 Polar coordinates

Keywords: *polar coordinates, Cartesian coordinates, polar coordinates, polar curves, Cartesian-to-polar method*

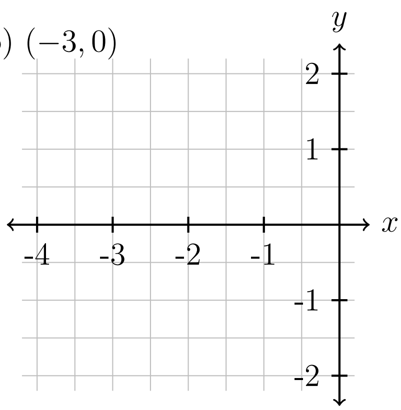
1. For each of the following conversions plot the point and convert.

Convert from Cartesian to Polar coordinates:

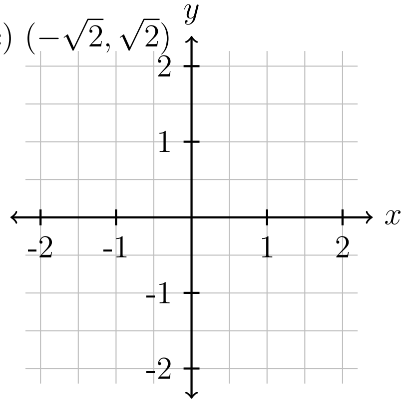
(a) $(0,2)$



(b) $(-3,0)$

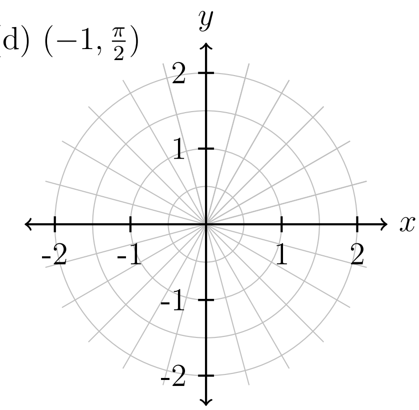


(c) $(-\sqrt{2},\sqrt{2})$

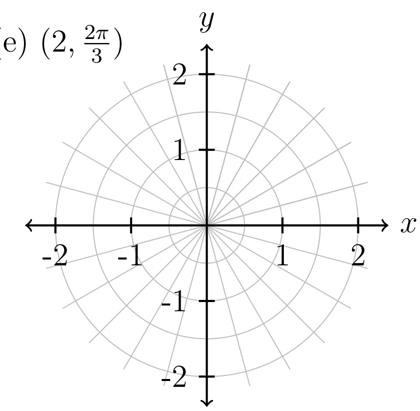


Convert from Polar to Cartesian coordinates:

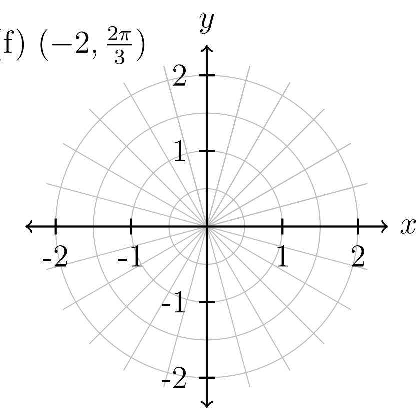
(d) $(-1,\frac{\pi}{2})$



(e) $(2,\frac{2\pi}{3})$



(f) $(-2,\frac{2\pi}{3})$

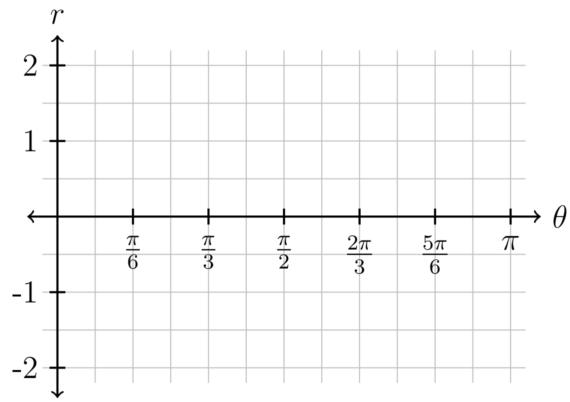


2. Consider the polar curve given by the equation $r = 2\cos(3\theta)$ with $0 \leq \theta \leq \pi$.

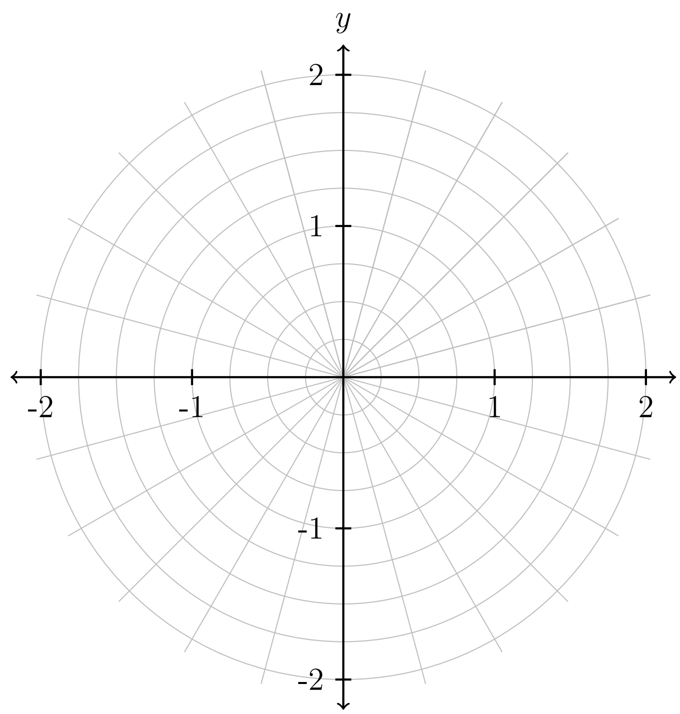
(a) Fill in the values of r in the table below.

θ	r
0	
$\pi/6$	
$\pi/3$	
$\pi/2$	
$2\pi/3$	
$5\pi/6$	
π	

(b) Sketch the curve in the θr-plane.



(c) Sketch the curve in the xy-plane. Label the points from the table.

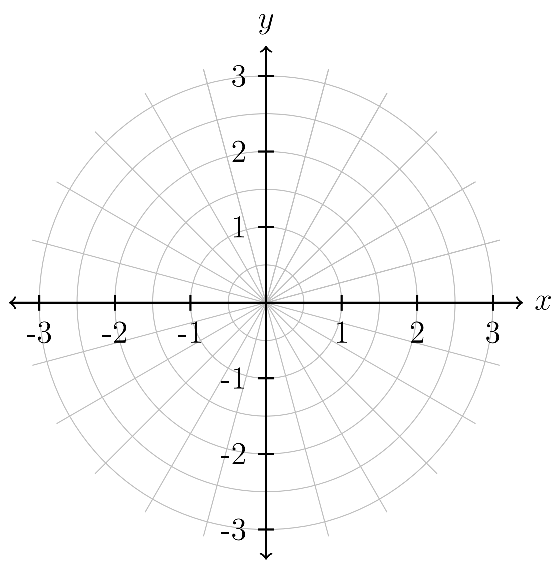


3. Consider the polar functions $r_1(\theta) = \dfrac{\theta}{\pi}$ and $r_2(\theta) = -\dfrac{\theta}{\pi}$ defined on $0 \le \theta \le 4\pi$.

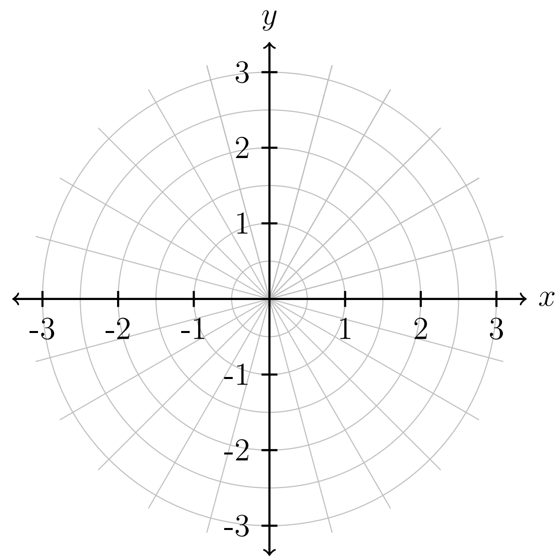
(a) Fill in the table below.

θ	$r_1(\theta)$	$r_2(\theta)$
0		
$\pi/2$		
π		
$3\pi/2$		
2π		
$5\pi/2$		
3π		

(b) Sketch $r_1(\theta)$ in the plane.



(c) Sketch $r_2(\theta)$ in the plane.



(d) How do the two graphs differ? What does multiplying a polar function by -1 do to it graphically in the xy-plane?

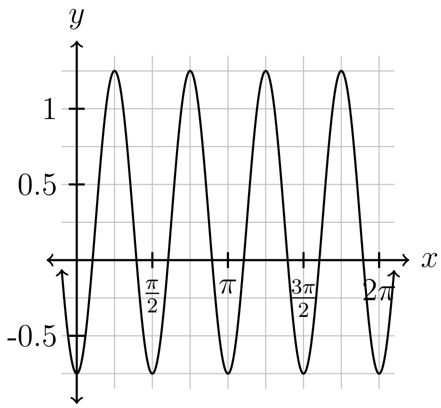
4. Match each of the equations (a)-(c) with one of the Cartesian graphs (i)-(iii) and one of the Polar graphs (I)-(III). Explain your reasoning.

   (a) $r = \frac{1}{4} - \cos(4\theta)$

   (b) $r = \sin(1 + 3\cos(\theta))$

   (c) $r = 1 - 2\sin(3\theta)$

(i)



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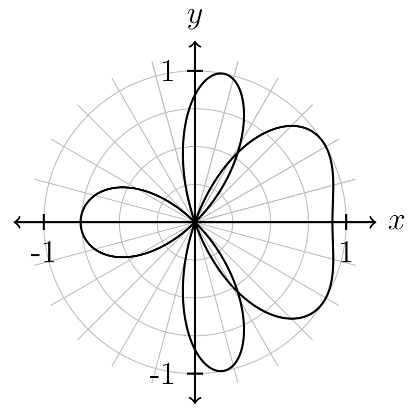

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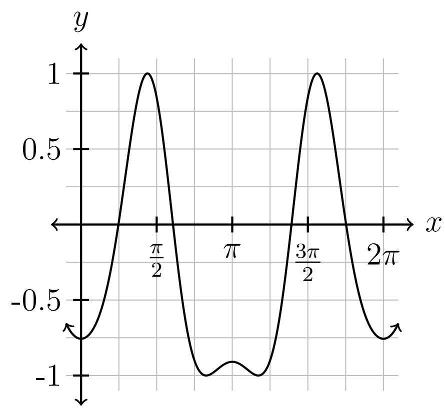

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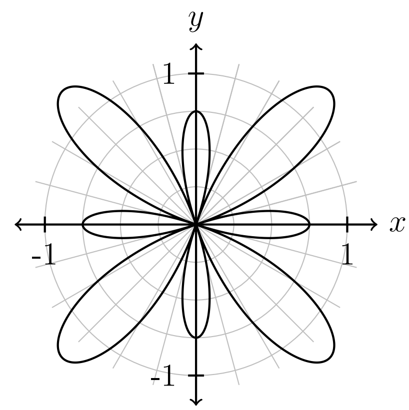

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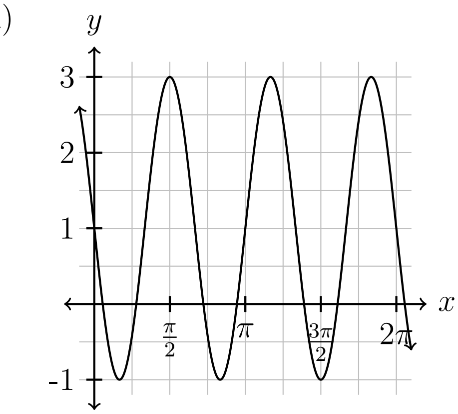

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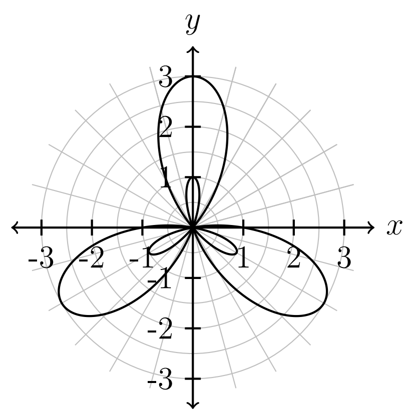


21 Linear systems

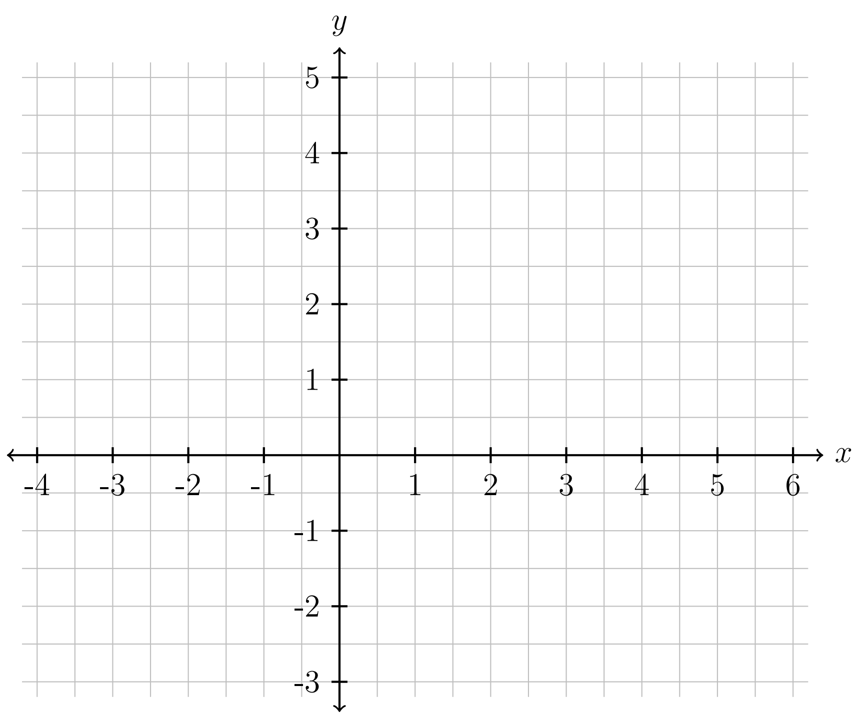
Keywords: *linear systems, system of equations, matrix, matrix multiplication, linear systems, augmented matrix*

1. We're going to look at the system of linear equations

$$\begin{aligned} 3x - 2y &= 1 \\ 2x - y &= 1. \end{aligned}$$

   (a) Solve this system of equations using algebra.

   (b) Draw the graphs of the equations for the two lines. Where do they intersect?



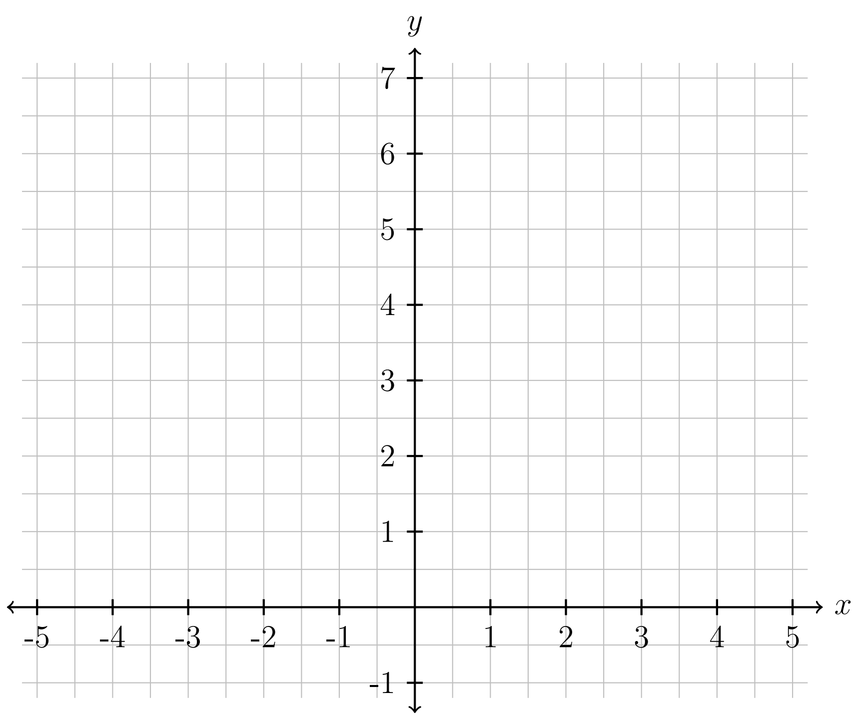
2. The system of equations

$$x + 2y = 6$$
$$2x + 4y = 13$$

has no solutions. What happens when you try to solve the system using:

(a) algebraic manipulations?

(b) a graph of the lines?



3. For what values of a does the system of equations $2x - y = a$, $4x - 2y = 3$ have a solution. How do the graphs of the lines change when you change the value of a?

4. For what values of a does the system of equations $2x - y = a$, $ax + y = 1$ have a solution? How do the graphs of the lines change when you change the value of a?

5. A movie made \$15 million online from a mix of sales and rentals. A sale costs \$16, and a rental costs \$5. Given that there were a total of 2 million transactions, how many were rentals and how many were sales?

6. You are trying to mix up 2 cups of a cinnamon-sugar topping consisting of 4/5 sugar and 1/5 cinnamon. You already have a large bag that is half sugar and half cinnamon, and a large bag full of pure sugar. How much should you take from each bag to get the correct mix?

22 Matrices

Keywords: *inverse matrix, determinants, linear systems*

1. Give an example of a:

   (a) 2×3 matrix with real entries.

$$A = \begin{bmatrix} \pi & 4 & -e \\ \frac{3}{2} & 0 & 1 \end{bmatrix}$$

   (b) 3×1 matrix with real entries.

$$B =$$

   (c) 2×3 matrix with integer entries.

$$C =$$

   (d) 3×3 matrix with integer entries.

$$D =$$

2. Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ -2 & 1 \end{bmatrix}.$$

   (a) Compute $3A$ and $B - 3A$. Can you find a matrix C that satisfies $3A - B + C = 0$?

   (b) Compute AB and BA.

3. Let A be the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

   Compute A^2.

4. Let
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 1 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 6 \\ 0 & 3 \end{bmatrix}.$$
Check that $AB = AC$, even though $B \neq C$.

5. The previous worksheet considered the system of equations
$$\begin{aligned} 3x - 2y &= 1 \\ 2x - y &= 1 \end{aligned}$$
   (a) Write down the augmented matrix corresponding to the system of equations, and use elimination to solve for x.

   (b) For each step, write down the system of linear equations corresponding to your matrix.

   (c) Does this match your answer from last time?

6. Consider the following augmented matrix:

$$\left[\begin{array}{rr|r} 1 & -3 & 4 \\ -2 & 6 & -8 \end{array}\right]$$

(a) What is the corresponding system of linear equations?

(b) Solve this system using elimination.

(c) What happens if the third column is replaced by $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$?

23 More matrices

Keywords: *inverse matrix, determinants, linear systems*

1. Find the determinants of the following matrices:

   (a) $A = \begin{bmatrix} -1 & 1 \\ -1 & 3 \end{bmatrix}$

   (b) $B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

   (c) $C = \begin{bmatrix} 2 & a \\ 0 & 3 \end{bmatrix}$

   (d) $D = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

2. Consider the matrix
$$C = \begin{bmatrix} 4 & 9 \\ 1 & 2 \end{bmatrix}.$$

   (a) Compute C^{-1}.

   (b) Double-check your answer by multiplying out CC^{-1} and $C^{-1}C$.

   (c) Use the first part to solve the linear system $C\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

3. In the previous worksheet, we looked at the system of equations

$$\begin{aligned} 3x - 2y &= 1 \\ 2x - y &= 1. \end{aligned}$$

(a) How can you write this system in matrix form $A\mathbf{x} = \mathbf{b}$?

(b) What is the inverse of the matrix A? Double-check your answer by multiplying out AA^{-1} and $A^{-1}A$. What answer should you get?

(c) Solve the linear system by multiplying both sides on the left by A^{-1}. Does this match your previous answer?

4. Last week, we saw that the system

$$\begin{aligned} x + 2y &= 6 \\ 2x + 4y &= 13 \end{aligned}$$

does not have any solutions.

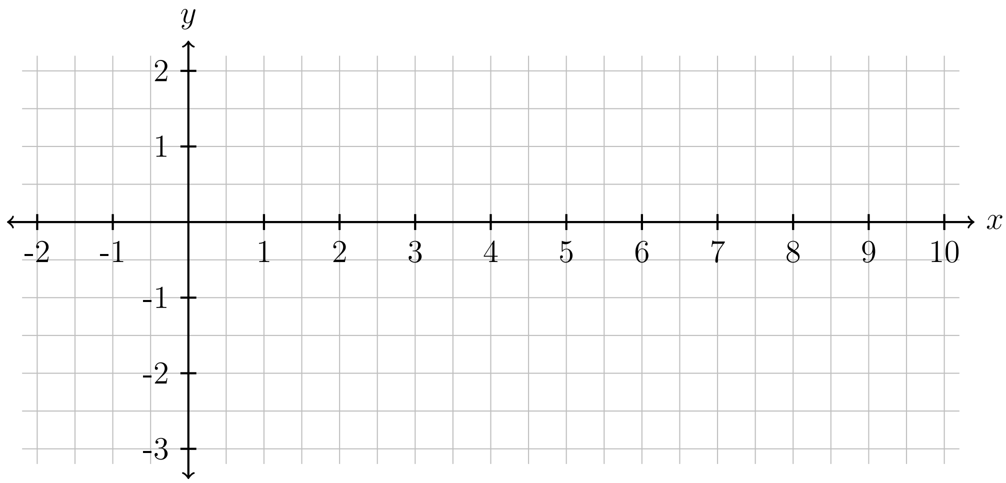
(a) Write this system of equations in matrix form.

(b) What happens when you try to solve by multiplying by A^{-1}?

24 Linear Maps

Keywords: *vectors, eigenvectors, eigenvalues, linear maps, rotations, identity map*

1. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Compute and graph the vectors $2\mathbf{x}$, $3\mathbf{y}$, and $2\mathbf{x} + 3\mathbf{y}$.



2. Consider the transformation $T(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

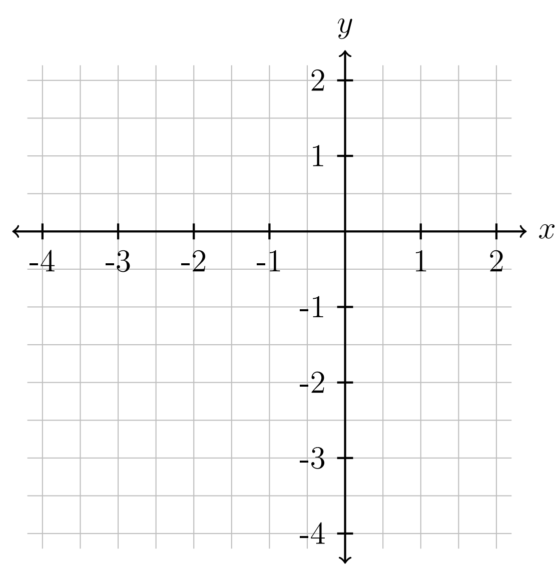
   (a) Plot $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, and their images under the transformation T. Describe geometrically what T does to each vector in the xy-plane.

   (b) Repeat the previous exercise for the transformations

$$S(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

   and

$$R(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$



3. Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(a) Compute $A\mathbf{x}$, where $\mathbf{x}$ is the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. What about $3\mathbf{x}$?

(b) Compute $A\mathbf{y}$, where $\mathbf{y}$ is the vector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

(c) What is the vector $\mathbf{x} - 2\mathbf{y}$? Check that $A(\mathbf{x} - 2\mathbf{y}) = A\mathbf{x} - 2A\mathbf{y}$.

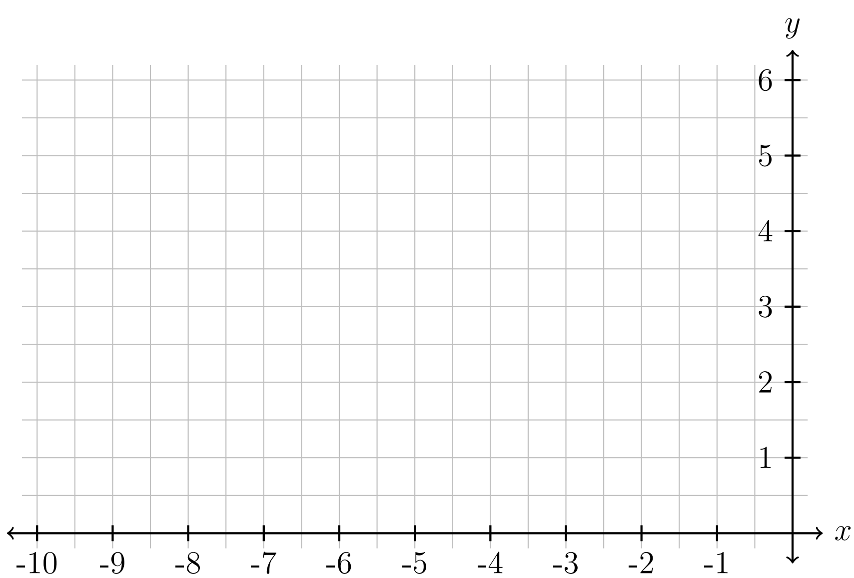
(d) Find a vector $\mathbf{x}$ so that $A\mathbf{x} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$.

4. Can you come up with a matrix A so that $A\mathbf{x}$ is the vector $\mathbf{x}$ rotated by 60 degrees counterclockwise? Check your answer by applying it to a vector of your choice.

5. Let $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

   (a) Compute $A\mathbf{x}$ and $A\mathbf{y}$.

   (b) Draw the four vectors $\mathbf{x}$, $\mathbf{y}$, $A\mathbf{x}$, and $A\mathbf{y}$ together on the coordinate plane. One of $\mathbf{x}$ and $\mathbf{y}$ is an eigenvector of A – which one? What is the corresponding eigenvalue?



6. (a) Let
$$B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$
What are the eigenvectors and eigenvalues for B? Try figure this out without making any calculations.

(b) Let
$$C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$
Can you find any eigenvalues for C? Why or why not? (Hint: geometrically, what does the matrix C do to a vector?)

(c) Let
$$D = \begin{bmatrix} 7 & -1 \\ 6 & 2 \end{bmatrix}.$$
Find all the eigenvalues and eigenvectors for D.

1 Review for first midterm

1. Let $g(x) = \displaystyle\int_0^x \frac{\cos(t)}{t}\,dt$.

   (a) Is $g(x)$ increasing or decreasing on $0 < x < \dfrac{\pi}{2}$?

   (b) What about on $\dfrac{\pi}{2} < x < \pi$?

2. Compute the following derivatives.

   (a) $\displaystyle\frac{d}{dx}\int_0^{x^3} \frac{\sqrt[3]{t}}{\ln(t)}\,dt$

   (b) $\displaystyle\frac{d}{dx}\int_{x^2}^{0} \frac{dt}{t}$

3. Compute the following integrals.

(a) $\displaystyle\int_0^2 \frac{4x}{x^2+2}\,dx$

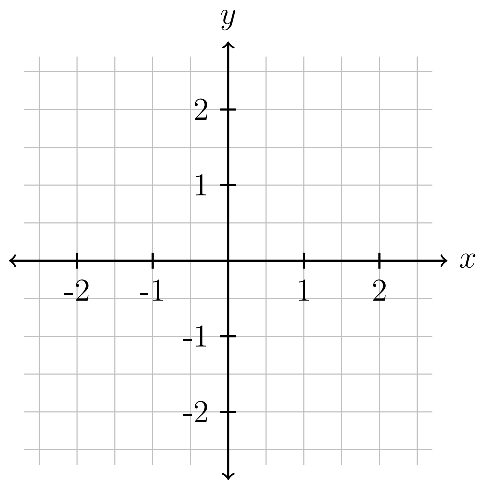
(b) $\displaystyle\int x^2 \ln(x^2)\,dx$

(c) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\sin(3x)}{1+\cos(3x)}\,dx$

(d) $\displaystyle\int x(\ln(x))^2\,dx$

4. Let R be the region bounded by $y = 1 - x^4$ and $y = |x| - 1$.

   (a) Sketch the region R.

   (b) Compute the area of R.

   (c) Set up the integral representing the length of the top edge of R. (You don't need to evaluate it.)

   (d) Set up the integral representing the volume of the solid obtained by rotating the region R about the line $y = -2$.

5. A parabolic antenna is formed by rotating the part of the graph of $y = x^2$ between $x = 0$ and $x = 3$ around the y-axis.

   (a) Draw a sketch of the antenna and compute its volume.

   (b) A rainstorm fills the antenna with water to a height of 5 m. Assuming water has a density of 1000 kg/m^3 and the acceleration due to gravity is 9.8 m/s^2. Set up the integral representing the work required to pump the water out over the top of the antenna.

6. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{2x}{x^2 - 3x - 10}\,dx$

   (b) $\displaystyle\int \frac{3x - 4}{x^2 - 3x + 2}\,dx$

   (c) $\displaystyle\int \sin^2(x)\,dx$

   (d) $\displaystyle\int \sin^2(x)\cos^3(x)\,dx$

7. Determine whether each the following improper integrals converge or diverge.

(a) $\displaystyle\int_1^{\infty} \frac{4\,dx}{e^x}$

(b) $\displaystyle\int_1^{2} \frac{4x+2}{x^2+x-2}\,dx$

(c) $\displaystyle\int_0^{\frac{\pi}{2}} \tan(\theta)\,d\theta$

2 Review for second midterm

1. Determine if the following sequences converge or diverge. If they converge, find the limit.

   (a) $\left\{-\dfrac{\sin(n)}{n}\right\}_{n=1}^{\infty}$

   (b) $\left\{\dfrac{(-1)^k}{\sqrt{k-3}}\right\}_{k=4}^{\infty}$

   (c) $\left\{\dfrac{n}{\sqrt{n-2}}\right\}_{n=3}^{\infty}$

   (d) $\left\{5+\dfrac{k^2+5}{2k^2+k-1}\right\}_{k=0}^{\infty}$

2. Determine if the following series converge or diverge. If they converge, find the sum.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n+1}}{5^n}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+3)(n+2)}$

   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{n+1}{n-1}$

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{n+1}{n^2-1}$

3. Determine if the following series converge absolutely, converge conditionally, or diverge.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-3)^k}{k!}$

   (b) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k \sqrt{k}}{1+k^2}$

   (c) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k+1}}$

4. Determine the radius and interval of convergence of the following power series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n4^n}$

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k(x-3)^k}{k^2}$

5. Use a Taylor polynomial of degree 2 to approximate $\sqrt{5}$.

6. Find a power series representation for $\dfrac{x^3}{x+2}$ and determine its radius and interval of convergence.

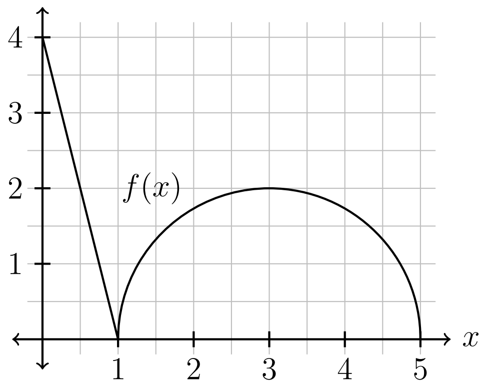
7. Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$.

   (a) Find a power series representation for $g(x) = x^2 f(-x^2)$.

   (b) Use the first three non-zero terms of the series you found in part (a) to approximate the definite integral $\int_0^1 g(x)\, dx$.

3 Review for final exam (part one)

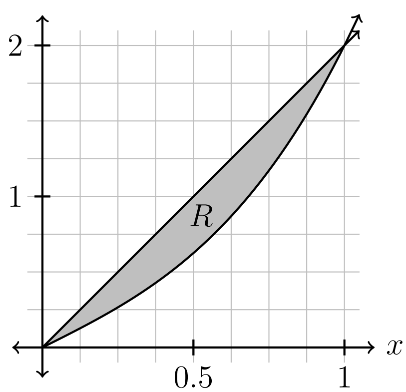
1. The graph of f below consists of a line segment, and a semi-circle of radius 2. Let $g(x) = \int_0^x f(t)\,dt$ for $0 \leq x \leq 5$.



Evaluate $g(0), g(1), g(5)$, and $g'(3)$.

2. Compute $\dfrac{d}{dx}\left(\int_1^{x^3} \dfrac{\sin t}{\sqrt{t}+1}\,dt\right)$.

3. Let R be the region enclosed by $y = x^3 + x$ and $y = 2x$ with $y \geq 0$.



(a) Set up the integral for the area of the region R.

(b) Set up the integral for the volume of the solid obtained by rotating R about the line $y = -1$.

4. Evaluate the following integrals.

   (a) $\displaystyle\int_0^1 (6x^2+8x+6)e^{x^3+2x^2+3x}\,dx$

   (b) $\displaystyle\int \frac{\sin(x)}{e^{\cos(x)}}\,dx$

   (c) $\displaystyle\int \frac{6x^2+4x-4}{x^3+x^2-2x}\,dx$

   (d) $\displaystyle\int_1^e x^2\,\ln(x)\,dx$

   (e) $\displaystyle\int \arctan(x)\,dx$

   (f) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^3(x)\,dx$

5. Find the arc length of the following curves on the given interval.

   (a) $y = \frac{2}{3}x^{\frac{3}{2}}$ on $[0, 3]$.

   (b) $y = 2\ln x - \frac{x^2}{16}$ on $[1, e]$.

6. Suppose an inverted conical tank with a height of 10 m and a radius of 1 m is filled completely with paint having a density of 1200 kg/m^3. Assume the acceleration due to gravity is 10 m/s^2.

   (a) What is the radius of the tank at a height of h meters?

   (b) Set up the integral representing the work required to empty the tank from the top.

7. For each of the following improper integrals, explain why it is improper and determine whether the integral converges or diverges.

   (a) $\displaystyle\int_1^3 \frac{dx}{(x-1)^2}$

   (b) $\displaystyle\int_1^\infty \frac{dx}{x^2+1}$

   (c) $\displaystyle\int_1^\infty \frac{4\,dx}{e^x}$

8. In a very wet period of summer, it persistently rained over several days. The precipitation rate P was recorded. It was measured in inches/hour, see the table given below for a snapshot of one full day. (a) Use Simpson's Rule, and (b) use the Trapezoidal Rule to estimate the total amount of precipitation that flooded the town on that day.

t	12 am	6 am	12 pm	6 pm	12 am
P	100	120	90	90	10

Use the following rules to estimate the total amount of precipitation that flooded the town on this day.

(a) Simpson's rule

(b) Trapezoidal rule

(c) Midpoint rule (Use two subintervals.)

9. Use the trapezoid rule with 3 sub-intervals to estimate $\int_0^3 x^3\,dx$.

4 Review for final exam (part two)

1. Determine whether the following statements are true or false. Explain the reasoning behind your answer.

   (a) The series $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} S_n = 5$ where $S_n = \sum_{k=1}^{n} a_k$.

   (b) If the series $\sum_{n=1}^{\infty} a_n$ converges, then the series $\sum_{n=1}^{\infty} |a_n|$ converges.

   (c) The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges.

   (d) If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

   (e) If the series $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

2. Determine whether the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges absolutely, conditionally or diverges. Justify you response. That is, state the name of any test you are using and verify that the conditions of the test are satisfied.

3. Determine whether the following series converge or diverge. If the series converges, compute its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \left(e^{1/n} - e^{1/(n+1)}\right)$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+1}{4n^2-3}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^n}{3^{2n+2}}$

4. Find the radius and the interval of convergence for the following power series.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-1)^k}{k3^k}$$

5. (a) Find the 3rd-degree Taylor polynomial of $f(x) = e^x$ centered at $x = 0$.

   (b) Use your results from part (a) to approximate $\sqrt[10]{e}$. You do not need to simplify your answer.

6. (a) Find a power series representation for $f(x) = \dfrac{1}{1+x^3}$.

   (b) Find a power series representation for $g(x) = \dfrac{x}{1+x^3}$.

   (c) Use your result in part (b) to approximate the indefinite integral $\displaystyle\int \frac{x}{1+x^3}\,dx$.

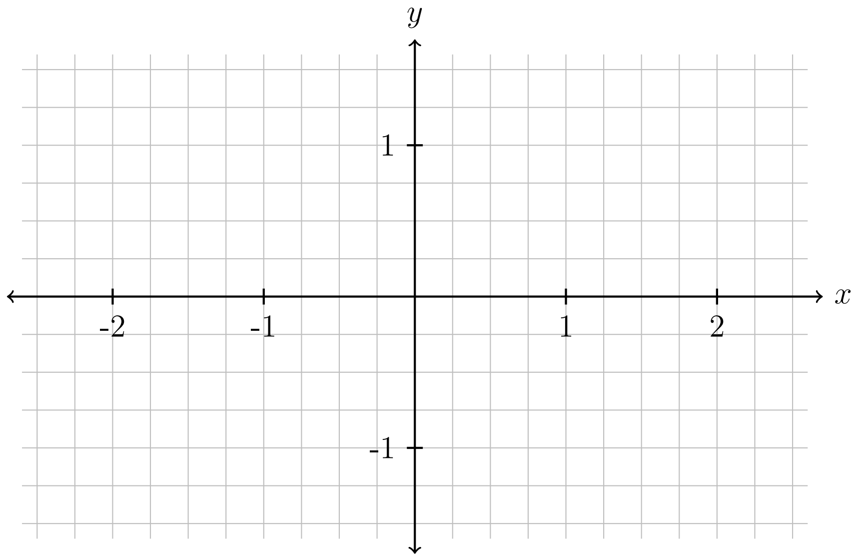
7. Let $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k^2} = x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \cdots$.

   (a) Find the first three non-zero terms of the power series representing the function $g(x) = f(x^4)$.

   (b) Use your results in part (a) to approximate the indefinite integral: $\displaystyle\int g(x)\,dx$.

8. Given the parametric equation $x = 2\cos(t)$ and $y = \sin(t)$ for $0 \leq t \leq 2\pi$.

   (a) Sketch a graph of the curve.



   (b) Find the slope of the tangent line at $t = \dfrac{3\pi}{4}$.

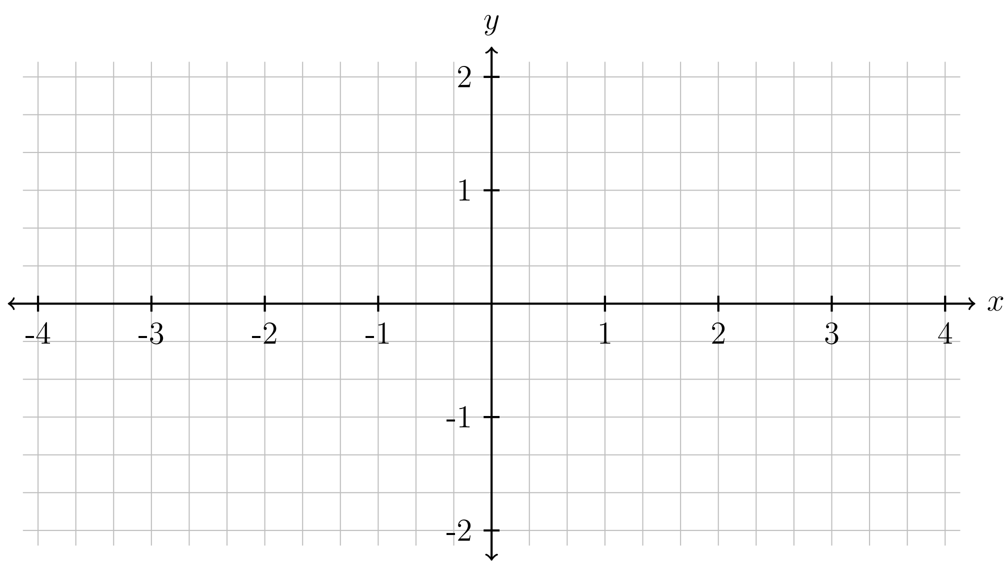
9. Find the Cartesian coordinates of the point on the polar curve $r = 2 + 4\cos(\theta)$ at $\theta = \dfrac{2\pi}{3}$.

10. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(a) Compute $A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3$, and $A\mathbf{v}_4$.

(b) Which of these vectors are eigenvectors of A? What are their corresponding eigenvalues?

(c) Sketch the eigenvectors and their corresponding images under the matrix A.



(d) Solve for $\mathbf{x}$ in $A\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ by first finding A^{-1}.

Index

absolute convergence, 45
alternating series test, 45
antiderivative, 4
applications, 45
approximation, 45
arc length, 19
arcs, 59
area, 13, 16
area of compound regions, 13
area under curve, 1
augmented matrix , 66
average value, 7

Cartesian coordinates, 62
Cartesian-to-polar method, 62
circles, 59
circular motion, 59
combining power series, 49
comparison test, 42
conditional convergence, 45
convergence, 33, 36

definite integral, 1, 4, 23
determinants, 69, 72
differentiating power series, 49
divergence, 33, 36
divergence test, 39

eigenvalues, 76
eigenvectors, 76
ellipses, 59
even functions, 7
exponential functions, 10

factoring, 29
Fundamental Theorem of Calculus, 4

geometric series, 36, 39

harmonic series, 39

identity map, 76
improper integral, 29
indefinite integral, 4, 23
infinite intervals, 29
infinite limit of integration, 29
infinite series, 36
integrating power series, 49
integration, 1, 4, 7, 10
integration by parts, 23, 26
interval of convergence, 49
inverse matrix, 69, 72

length of curves, 19
limit comparison test, 42
limits of sequences, 33
linear approximation, 45
linear maps, 76
linear systems, 66, 69, 72
long division, 29

Maclaurin series, 56
matrix, 66
matrix multiplication, 66
midpoint rule, 53

numerical integration, 53

odd functions, 7

p-test, 39
parametric equations, 59
partial fractions, 29
partial sums, 36
physical applications, 19
polar coordinates, 62
polar curves, 62
polynomials, 29
power series, 45, 49
pumps, 19

quadratic approximation, 45

radius of convergence, 49
ratio-test, 39
recursive sequences, 33
region between curves, 13, 16
Riemann sum, 1
rotations, 76

sequences, 33
series, 36, 39, 42, 45
series approximation, 56
series expansion, 56
Simpson's rule, 53
slicing method, 13, 16
slope, 59
solid of revolution, 16
springs, 19
substitution, 10
symmetric functions, 7
symmetry, 7
system of equations, 66

Taylor polynomials, 45
Taylor series, 56
telescoping series, 36
trapezoid rule, 53
trigonometric functions, 10, 26
trigonometric integration, 26

unbounded integrands, 29

vectors, 76
volume, 13, 16

work, 19