Algebra Notes – Math 435 – Spring 2014

Weeks 1 and 2.
Homework on page 17.

LK

I. Review quad eqn & cx nos.

$ax^2+bx+c=0$, $a \neq 0$ & $a>0$.

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

$$\left(x^2+2\left(\frac{b}{2a}\right)x+\left(\frac{b}{2a}\right)^2\right)-\left(\frac{b}{2a}\right)^2+\frac{c}{a}=0$$

$$\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}=0$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

Real solutions when $b^2-4ac \geq 0$.
Complex number (imaginary) solutions when $b^2-4ac<0$.

Simplest complex soln:

$$x^2+1=0$$

$$x=\left(-0\pm\sqrt{0^2-4(1)(1)}\right)/2$$

$$x=\pm\sqrt{-1}.$$

There is no real number $\sqrt{-1}$.
Initially people dealt with $\sqrt{-1}$ by writing $i=\sqrt{-1}$, $-i=-\sqrt{-1}$ and using the rule $i^2=-1$.

Then, assuming i behaves otherwise as an ordinary number, we have

$$\begin{aligned}(a+bi)(c+di) &= a(c+di) + bi(c+di)\\ [a,b,c,d \text{ real}] \quad &= ac + adi + bci + bdi^2\\ &= (ac-bd) + (ad+bc)i\end{aligned}$$

This actually gives us a rigorous definition of complex nos. (Due to W. R. Hamilton in 1800's). We identify $a+bi = (a,b)$ as an ordered pair of real numbers. Thus $1 = (1,0)$

$$i = (0,1).$$

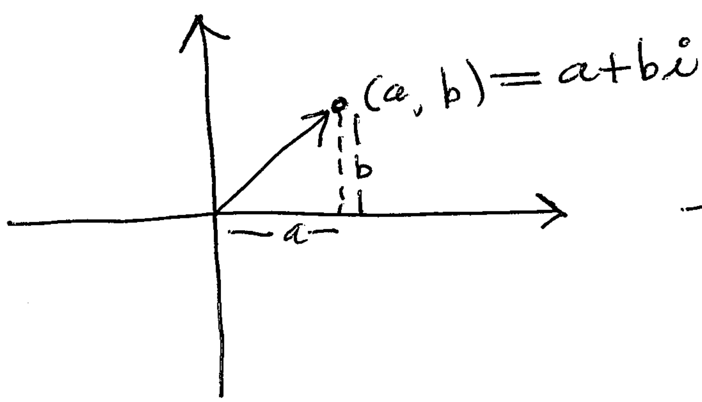
and define the product of these pairs by: $(a,b)(c,d) = (ac-bd, ad+bc)$

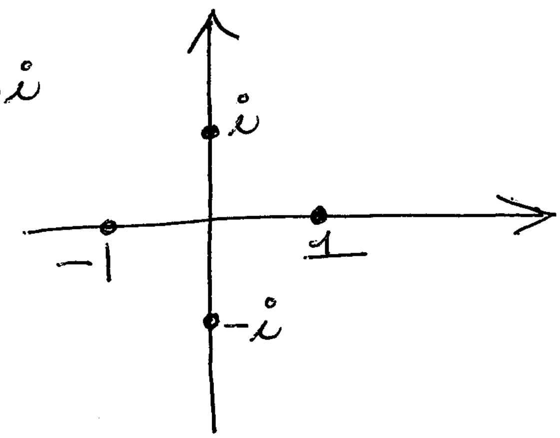
and the sum by: $(a,b)+(c,d) = (a+c, b+d)$.

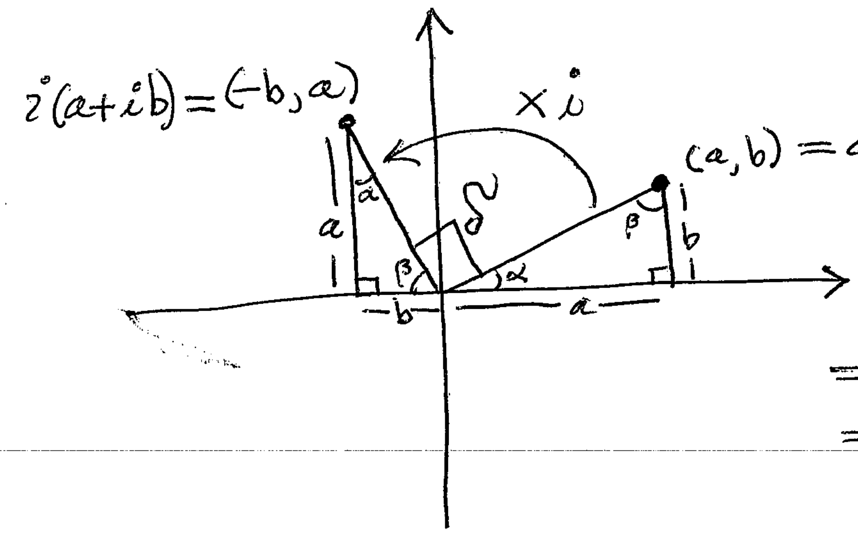
In this way we have $(a,b) = (a',b')$ if and only if $a=a'$ and $b=b'$.

Note that $(a,b)i = (-b,a)$.

If you think of (a,b) as a point in the plane, then we have a geometric interpretation of complex numbers.







right $\Delta \Rightarrow \alpha+\beta = \pi/2$

$\alpha+\beta+\delta =$ straight $\measuredangle$

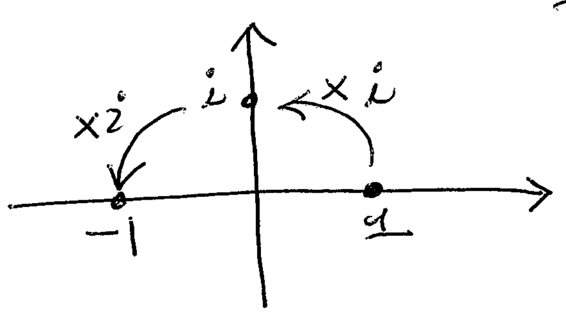
$\Rightarrow \alpha+\beta+\delta = \pi$

$\Rightarrow \delta = \pi - (\alpha+\beta)$

$\delta = \pi - \pi/2$

$\therefore \delta = \pi/2$

Multiplication by i rotates a point in the plane by $\pi/2 = 90^\circ$.



$i^2 = -1$ <u>means</u> that two 90° rotations make a 180° rotation!

<u>Theorem</u>. Let $Z_\theta = \cos(\theta) + i\sin(\theta)$.

Then $Z_\theta Z_\phi = Z_{(\theta+\phi)}$.

<u>Proof</u>. $Z_\theta Z_\phi = [c(\theta) + i\,s(\theta)][c(\phi) + i\,s(\phi)]$

$= [c(\theta)c(\phi) - s(\theta)s(\phi)] + i[s(\theta)c(\phi) + c(\theta)s(\phi)]$

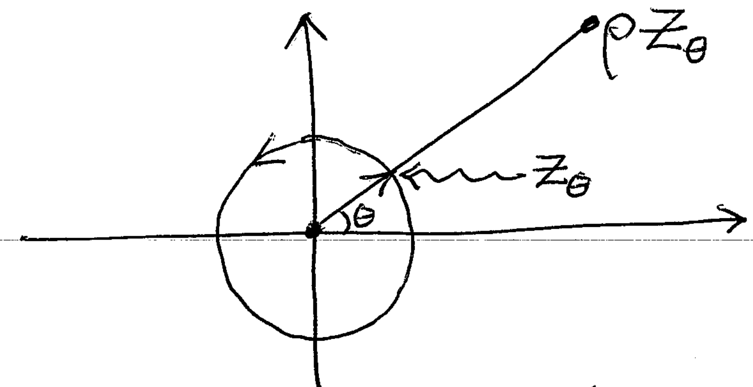
$= c(\theta+\phi) + i\,s(\theta+\phi)$

$= Z_{(\theta+\phi)}$ //

$s(\theta) = \sin(\theta)$
$c(\phi) = \cos(\phi)$

We use basic trig formulas for sin & cos of sums of $\measuredangle$s.

Any complex number can be written in the form $Z = \rho Z_\theta$ for some $\rho \geqslant 0$ and θ an angle between 0 & 2π.



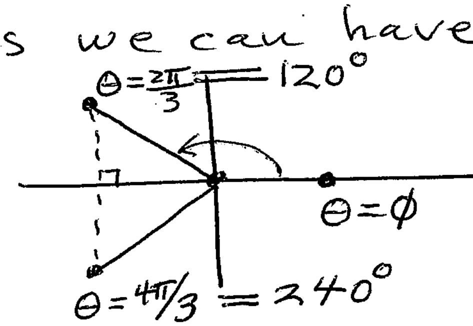
Thus, when you multiply complex nos, you <u>add their angles</u> and <u>multiply their lengths</u>.

$$(\rho Z_\theta)(\rho' Z_{\theta'}) = (\rho\rho') Z_{(\theta+\theta')}.$$

$\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$, $\mathbb{R}$ = the real numbers.

<u>Example</u>. Find all $Z \in \mathbb{C}$ s.t. $Z^3 = 1$.

<u>Answer</u>. Let $Z = Z_\theta$ then $3\theta = 2\pi N$. So $\theta = 2\pi N/3$. Thus we can have $\theta = 2\pi/3, 4\pi/3, \phi$:



$$\left.\begin{aligned} Z_{\frac{2\pi}{3}} &= -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i \\ Z_{4\pi/3} &= -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i \\ Z_\phi = 1 &= 1 \end{aligned}\right\}$$

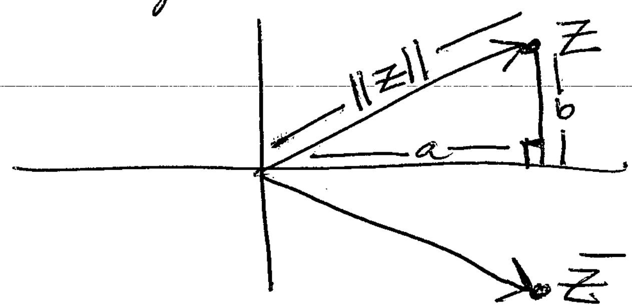
Let $\boxed{\begin{aligned} \omega &= \frac{-1+\sqrt{3}\,i}{2} \\ \omega^2 &= \frac{-1-\sqrt{3}\,i}{2} \end{aligned}}$

$\boxed{\text{$1, \omega, \omega^2$ are the three cube roots of unity.}}$

<u>Some facts.</u>

1°. $\begin{array}{l} z = a+bi \\ \bar{z} \underset{\text{def}}{=} a-bi \end{array} \Bigg\} \quad \begin{array}{l} z\bar{z} = (a+bi)(a-bi) \\ \underline{z\bar{z} = a^2+b^2} = \|z\|^2 \end{array}$

$$\|z\| \underset{\text{def}}{=} \sqrt{a^2+b^2} = \sqrt{z\bar{z}}.$$



$\|z\|^2 = a^2+b^2.$

2°. $\overline{zw} = \bar{z}\,\bar{w}.$

<u>Pf.</u> $\overline{zw} = \overline{(a+bi)(c+di)}$

$= \overline{(ac-bd)+(ad+bc)i}$

$= (ac-bd) - (ad+bc)i$

$\bar{z}\bar{w} = (a-bi)(c-di)$

$= (ac-bd) - (bc-ad)i$ ✓ //

3°. <u>Note</u>:

$(a^2+b^2)(c^2+d^2) = z\bar{z}w\bar{w}$

$= (zw)(\bar{z}\bar{w})$

$= (zw)\overline{(zw)}$

$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$

Thus <u>the product of two sums of squares is a sum of squares</u>. (Try $(1^2+2^2)(3^2+4^2)$.)

4. Solving cubic equations.

THE WHOLE STORY

Fact: $(a+b)^3 = a^3+b^3+3ab^2+3a^2b$

So $(a+b)^3 = (3ab)(a+b) + (a^3+b^3)$.

If you were asked to solve

$$X^3 = pX + q$$

you can set $X = a+b$ (a, b unknown) and try to find a, b s.t.

$$\boxed{\begin{aligned} p &= 3ab \\ q &= a^3+b^3 \end{aligned}}.$$

This leads to a quadratic equation for $\boxed{a^3 = R}$ and $\boxed{b^3 = S}$.

$$(p/3)^3 = a^3b^3 = RS$$

$$q = a^3+b^3 = R+S.$$

Solve for R and S.
Then take their cube roots and add them up.

Example: $X^3 = 3X + 1$.

$p=3$, $q=1$

$$\boxed{\begin{aligned} 3 &= 3ab \\ 1 &= a^3+b^3 \end{aligned}} \quad \text{So} \quad \boxed{\begin{aligned} 1 &= ab \\ 1 &= a^3+b^3 \end{aligned}}$$

$$\text{So} \quad \left.\begin{aligned} 1 &= a^3b^3 = RS \\ 1 &= a^3\cancel{b}^3 = R+S \end{aligned}\right\} \quad \boxed{\begin{aligned} 1 &= RS \\ 1 &= R+S \end{aligned}}$$

next page

$$\boxed{\begin{aligned} 1 &= RS \\ 1 &= R+S \end{aligned}}$$

So $S = 1 - R$

&

$1 = RS = R(1-R)$

$$1 = R - R^2$$

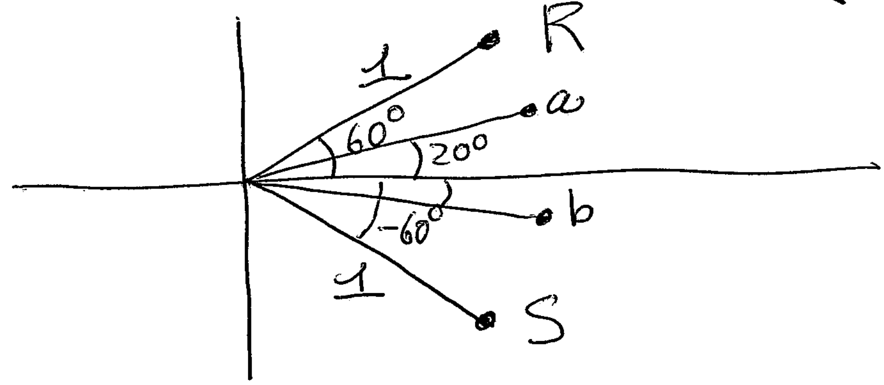
$$R^2 - R + 1 = 0$$

$$R = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}\,i}{2}.$$

This means we can take

$$a^3 = R = \frac{1+\sqrt{3}\,i}{2} = \cos(60^\circ) + i\sin(60^\circ)$$

$$b^3 = S = 1 - R = \frac{1-\sqrt{3}\,i}{2}.$$



So we can take $a = \cos(20^\circ) + i\sin(20^\circ)$

$b = \cos(20^\circ) - i\sin(20^\circ)$

$$a = \sqrt[3]{\frac{1+\sqrt{3}\,i}{2}}, \quad b = \sqrt[3]{\frac{1-\sqrt{3}\,i}{2}}$$

These are the "principal cube roots of R and S".

Note that $X = a + b$ is <u>real</u>.

$$X = 2\cos(20^\circ).$$

Thus we have shown that $X^3 = 3X + 1$ has a <u>real root</u> $= \underline{2\cos(20^\circ)} = \sqrt[3]{\frac{1+\sqrt{3}i}{2}} + \sqrt[3]{\frac{1-\sqrt{3}i}{2}}$.

What about the other roots of $X^3 = 3X + 1$?

Recall our 3 cube roots of unity $\{1, \omega, \omega^2\}$. We have

$$ab = \sqrt[3]{R}\,\sqrt[3]{S} = \sqrt[3]{\left(\frac{1+\sqrt{3}i}{2}\right)\left(\frac{1-\sqrt{3}i}{2}\right)} = \sqrt[3]{1} = 1.$$

If we replace a by $\omega a = a'$

b by $\omega^2 b = b'$

then $a'b' = \omega a \omega^2 b = \omega^3 ab = 1.$

So we still have

$$p = 3a'b'$$

$$\& \quad q = a'^3 + b'^3.$$

Thus we have found <u>3</u> roots to the cubic:

$a + b$,

$\omega a + \omega^2 b$,

$\omega^2 a + \omega b$.

Are the two new roots complex or real?

A complex number z is real if and only if $z = \bar{z}$.

$\Big[z = a+bi;\ z = \bar{z} \iff a+bi = a-bi \iff a = a$ and $b = -b \iff b = 0 \Big]$ ✓

Now $a+b = a+\bar{a}$ $\quad (b = \bar{a})$

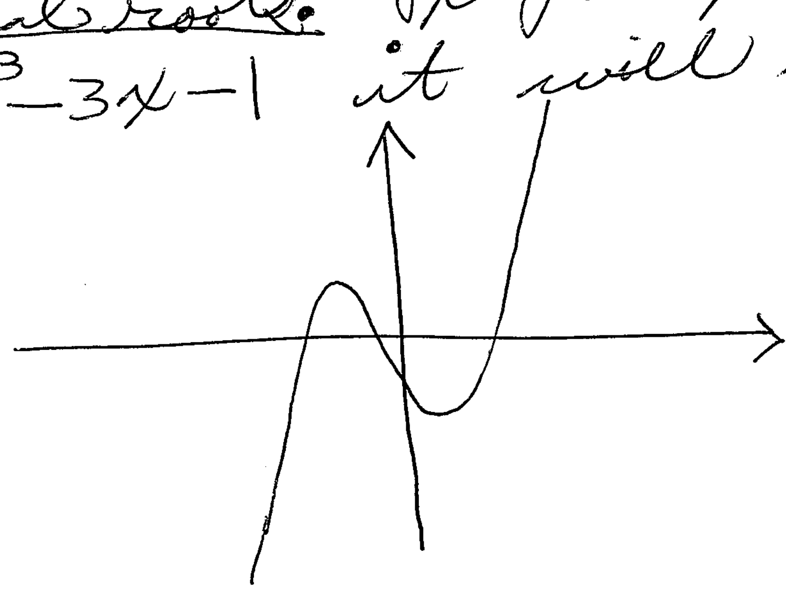
So $\overline{a+b} = \overline{a+\bar{a}} = \bar{a} + \bar{\bar{a}} = \bar{a} + a = b+a$
$= a+b$ ✓

$\Big[\bar{\bar{z}} = z$ for any z
$\overline{z+w} = \bar{z} + \bar{w}. \Big]$

Note: $\boxed{\bar{\omega} = \omega^2 \quad \overline{\omega^2} = \omega \quad \& \quad \bar{a} = b}$

But $\overline{\omega a + \omega^2 b} = \bar{\omega}\bar{a} + \overline{\omega^2}\,\bar{b}$
$= \omega^2 b + \omega a = \omega a + \omega^2 b.$

Thus the other two roots of $X^3 = 3X+1$ are <u>also real!</u> and so $X^3 = 3X+1$ has <u>three real roots.</u> If you plot $y = x^3 - 3x - 1$ it will look like:



Here is another example.

Consider the cubic

$$X^3 = -3X + 1$$

Then with $X = a+b$, we have

$$\boxed{\begin{array}{l} +3ab = -3 \\ a^3+b^3 = 1 \end{array}}$$

$$\Downarrow$$

$$\boxed{\begin{array}{l} ab = -1 \\ a^3+b^3 = 1 \end{array}} \Rightarrow \boxed{\begin{array}{l} a^3b^3 = -1 \\ a^3+b^3 = 1 \end{array}}$$

So $\boxed{\begin{array}{l} RS = -1 \\ R+S = 1 \end{array}}$

$$R(1-R) = -1$$

$$R - R^2 = -1$$

$$R^2 - R - 1 = 0$$

$$R = \frac{1 \pm \sqrt{5}}{2}$$

The quadratic has real roots.

So $a^3 = \frac{1+\sqrt{5}}{2}$, $b^3 = \frac{1-\sqrt{5}}{2}$

and the full set of solns to the cubic is: (next page)

Let $a = \sqrt[3]{\dfrac{1+\sqrt{5}}{2}}$, $b = \sqrt[3]{\dfrac{1-\sqrt{5}}{2}}$

These are real numbers.

$a > 0$, $b < 0$.

Then solns to $x^3 = -3x + 1$

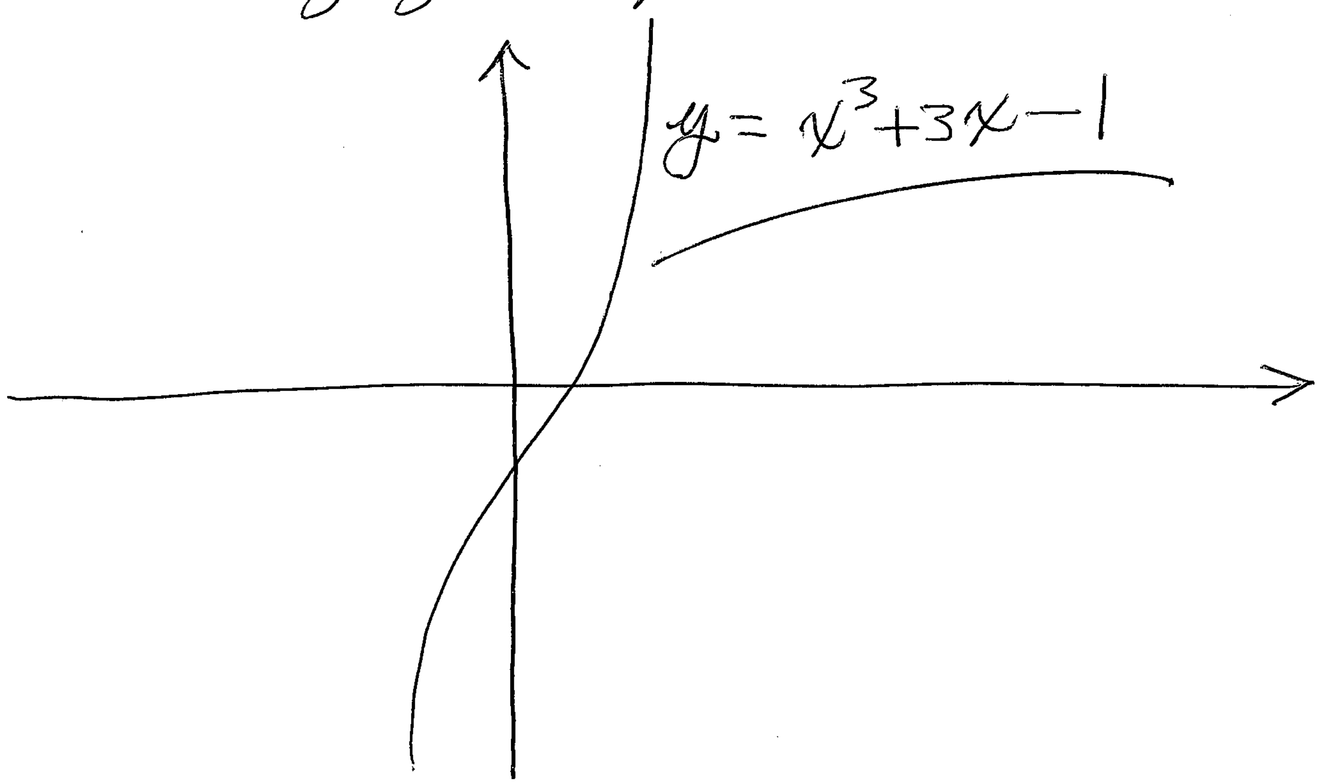
are: $a + b$ (real)

$\left.\begin{array}{l} \omega a + \omega^2 b \\ \omega^2 a + \omega b \end{array}\right\}$ both complex.

Note that

$$\overline{\omega a + \omega^2 b} = \bar{\omega}\,\bar{a} + \overline{\omega^2}\,\bar{b} = \omega^2 a + \omega b.$$

Thus the two complex roots are conjugates of each other.



II. Induction

Principle of Mathematical Induction

Suppose that $P(n)$ is a statement about a natural number n.
$n \in N = \{1, 2, 3, 4, \dots\}$.
Then $P(n)$ is true for all $n \in N$

if you prove:

I. $P(1)$ is true.

and II. If $P(k)$ is true (for some k) then $P(k+1)$ is true.
That is, you must show that $P(k) \Longrightarrow P(k+1)$.

Example. Show that

$$1+3+5+\dots+(2n-1) = n^2$$

for all $n = 1, 2, 3, \dots$.

Solution. I. $P(n): 1+3+\dots+(2n-1) = n^2$

$$P(1): \ 1 = 1^2 \quad (2\cdot 1 - 1 = 1)$$

$\therefore P(1)$ is true.

II. Suppose $1+3+\dots+(2k-1) = k^2$
(i.e. assume $P(k)$ is true).

Then $1+3+\dots+(2k-1)+(2(k+1)-1)$

$$= k^2 + 2(k+1) - 1$$
$$= k^2 + 2k + 1$$
$$= (k+1)^2.$$

Thus we showed that $P(k) \Longrightarrow P(k+1)$. //

Strong Induction. For part II you can assume that $P(1), P(2), \ldots, P(k)$ are all true and prove from that, that $P(k+1)$ is true.

[It is not hard to show that Induction and Strong Induction are logically equivalent.]

Example.

Theorem. Let $S \subseteq N = \{1, 2, 3, \ldots\}$ be a non-empty subset of the natural numbers N. Then S has a least member.

Proof. Let $P(n): n \notin S$.

Suppose that S has no least member and that $S \subseteq N$. We will prove, by strong induction, that S is empty!

I. $1 \notin S$ since $1 \in S \Rightarrow S$ has 1 as a least member.

II. Suppose that $1 \notin S, 2 \notin S, \ldots, k \notin S$. Then clearly if $(k+1) \in S$, $k+1$ would be the least member of S.
$\therefore k+1 \notin S$.

We have proved, by induction, that S is empty. $\therefore$ S not empty implies S has a least member. //

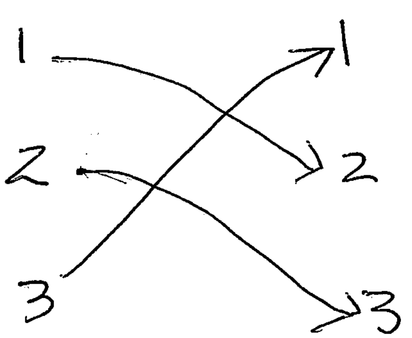
III. Multiplying Permutations

The permutations of 1,2,3 are:

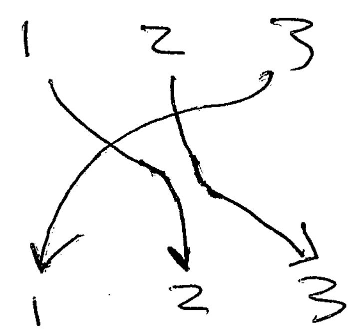
$$\left.\begin{array}{l} 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array}\right\}$$ These are all the ways to arrang three things (1, 2, 3) in order.

I will write $\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$ to denote a given permutation. Thus $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ denotes "231".

I think of this as a mapping from the set $\{1,2,3\}$ to itself.



or

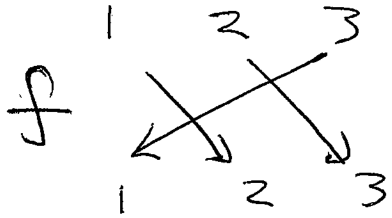


or

$$f : \{1,2,3\} \longrightarrow \{1,2,3\}$$

$$\left.\begin{array}{l} (1)f = 2 \\ (2)f = 3 \\ (3)f = 1 \end{array}\right\}$$ Notice I write $(x)f$ instead of $f(x)$!

Suppose



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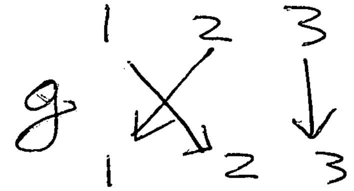
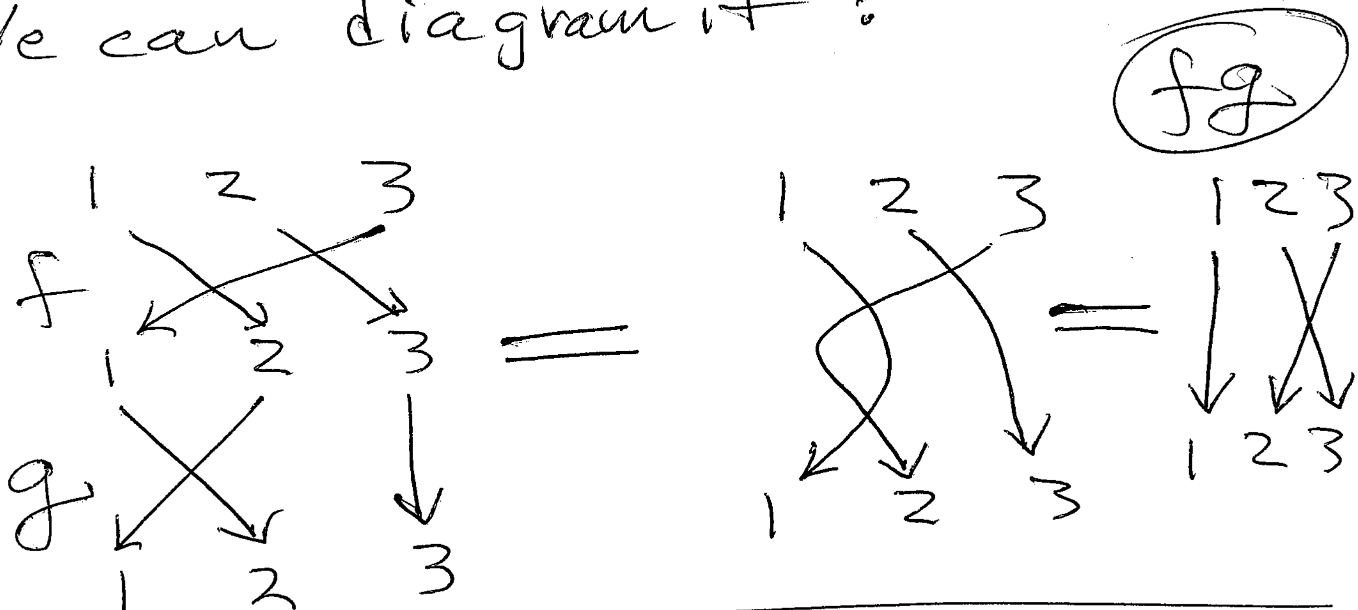


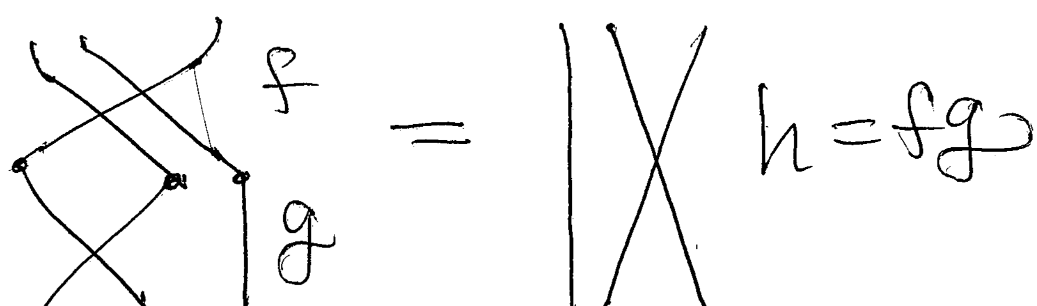
Then $(x)fg$ = result of doing f & then doing g.

e.g $(1)fg = (2)g = 1$.

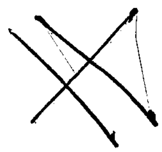
We can diagram it:

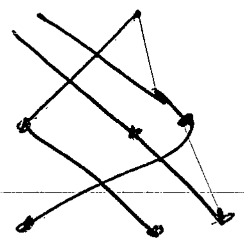
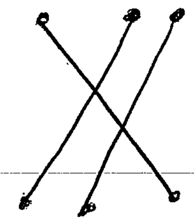


I will eliminate the arrows and write

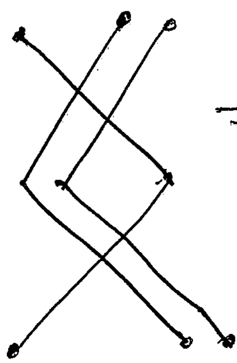
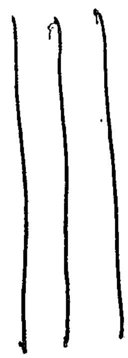


So you see, you can multiply permutations.

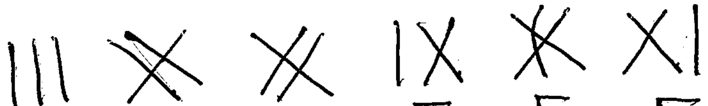
e.g. $R =$ 

$R^2 =$  $=$ 

simplify and redraw ~~the~~ connections. Only endpts matter.

$R^3 =$  $=$  $= I$.

We have 6 permutations.



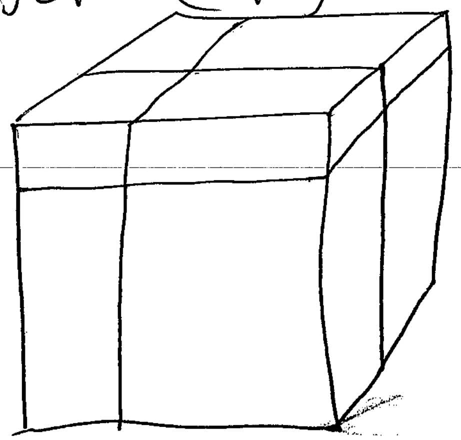
$I \quad R \quad R^2 \quad F_1 \quad F_2 \quad F_3$

You can make a 6×6 multiplication table.

(Homework.)

Homework

1. Draw the full architecture for $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$



Show how a cube of side $a+b$ decomposes into 3dim rectangular parts in corres with the above formula.

2. Prove by induction ~~that~~
$(1^3 + 2^3 + \cdots + n^3) = (1 + 2 + \cdots + n)^2$
for all $n = 1, 2, 3, \ldots$

3. Prove by induction (strong induction) that every natural number can be written as a sum of <u>distinct</u> powers of 2.
(e.g. $27 = 2^4 + 2^3 + 2^1 + 2^0$)

4. Using numbers different from our examples, solve a cubic of the form $X^3 = pX + q$
(p and q real numbers that you choose.)

5. Make a multiplication table for $\{I, R, R^2, F_1, F_2, F_3\}$ as on page 16.