1°, 2°, 3°, 4°, 5° Problems For this Week

Math 435 — Algebra Notes — Week 3 + Problems

LK

1° Every $n \in N = \{1, 2, \ldots\}$ is a sum of powers of 2.

<u>Proof.</u> $n = 2k \Rightarrow n = \underbrace{2 + 2 + \cdots + 2}_{k\ 2\text{'s}}$

and $2 = 2^1$.

$n = 2k+1 \Rightarrow n = \underbrace{2 + 2 + \cdots + 2}_{k\ 2\text{'s}} + \underset{\sim}{2^0}$ //

Of course, the better result is to show that $n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_m}$ where $a_1 > a_2 > a_3 > \cdots > a_m$.

But $2^k + 2^k = 2^{k+1}$, so if you have repetitions of powers of 2, you can combine them.

e.g. $2^3 + 2^3 + 2^2 + 2^2 + 1$

$= 2^4 + 2^3 + 1$.

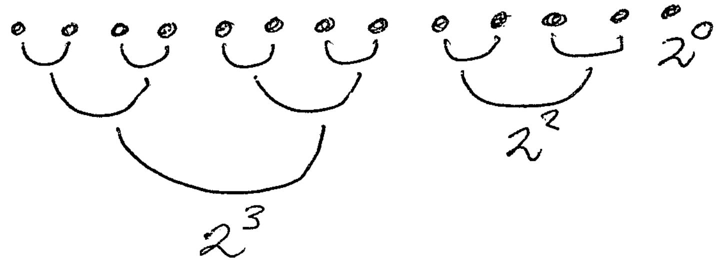
Or even,

$2 + 2 + 2 + 2 + 2 + 1$

$= 4 + 4 + 2 + 1$

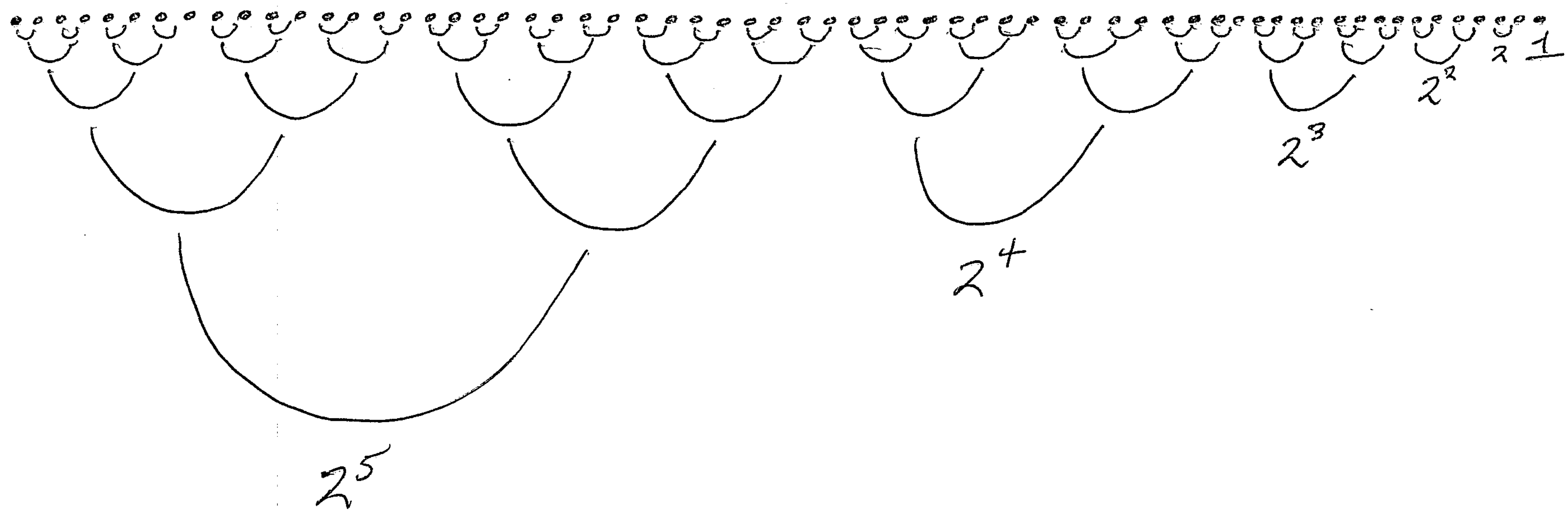
$= 8 + 2 + 1$ ✓

In fact,



//

1.1



$$2^5+2^4+2^3+2^2+2+1=2^6-1$$

Problem 1. (a) Prove, using induction, that $2^n > n$ for $n \in N = \{1, 2, 3, \ldots\}$.

(b) Prove that for $n \in N$, there exists a least $\ell = \ell(n) \in N$ such that $2^{\ell(n)} \geq n$.

(c) Using (b), show that for $n \in N$ there is a largest $K = K(n)$ such that $2^{K(n)} \leq n$.

(d) Using (c), prove that every $n \in N$ can be written as a sum of distinct powers of 2.

(e) Prove, by induction that
$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$$

(f) Discuss the following "argument".

Let $S = 1 + 2 + 2^2 + 2^3 + \cdots$,

then $2S = \quad 2 + 2^2 + 2^3 + \cdots$

$\therefore S = 2S - S = -1.$

So $-1 = 1 + 2 + 2^2 + 2^3 + \cdots$

In binary: $\begin{array}{r} 1111 \\ +\,1 \\ \hline 10000 \end{array}$ carry takes the 1 out to here.

In a computer with a limited register (e.g. 8 bits) $\boxed{1\,|\,1\,|\,1\,|\,1\,|\,1\,|\,1\,|\,1\,|\,1} + 1 \to \boxed{0\,|\,0\,|\,0\,|\,0\,|\,0\,|\,0\,|\,0\,|\,0}$ since the 9th bit is discarded.

2° Solve the cubic

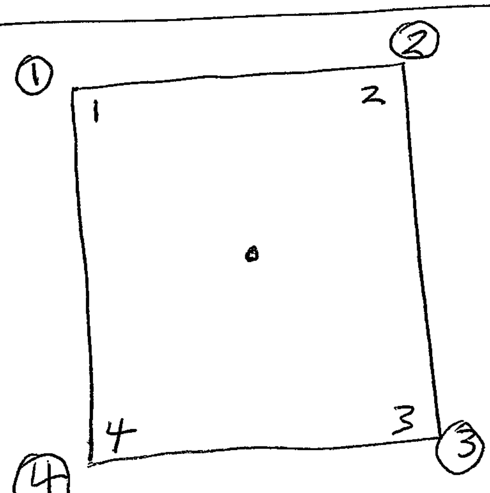
$$x^3 = 15x + 4$$

via our method of using an associated quadratic equation.

Compare your results to the fact that $4^3 = 15 \times 4 + 4$. From this you can show that

$$(x^3 - 15x - 4) = (x-4)(x^2 + 4x + 1)$$

and get direct solutions. Which of your solutions from the first method correspond to the solutions in the second method?

3° 

Make a catalog of the symmetries of a square. These should include the $\frac{2\pi}{4}$-rotation and the flips around diagonal, vertical and horizontal axes.

You can express your results as permutations in S_4. For example, the clockwise rotation by $2\pi/4$ is $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$.

The complete table should form a group. Investigate this group. How many elements does it have? What are its subgroups?

4. Matrix Multiplication and Groups

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} \underset{df}{=} \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

Let $R = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$,

$F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $F_3 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$.

Show:

(a) $R^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

(b) $F_1^2 = F_2^2 = F_3^2 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) $F_2 F_1 = R$

(d) $F_1 F_2 = R^2$

As you can see, the set of matrices $\{I, R, R^2, F_1, F_2, F_3\}$ forms a group with exactly the same properties as S_3 where we interpreted I, R, R^2, F_1, F_2, F_3
III XX XX IX XX XI.

Think about this!

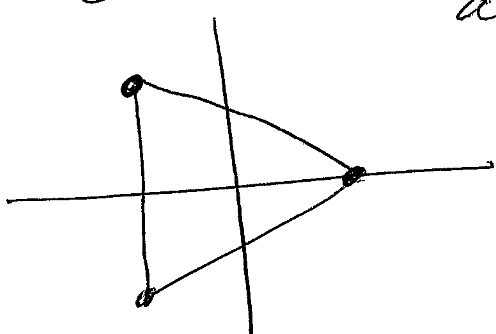
5.° Take a different tack to what happened in problem 4.° You know that S_3 is "the same" as symmetries of a triangle.

Write a matrix for a $\frac{2\pi}{3}$ rotation.

$$R_{\frac{2\pi}{3}} \leftrightarrow \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$$

$$\leftrightarrow \begin{bmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{bmatrix} = R$$

and a matrix for a flip around x-axis.



$$F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Show how F and R generate 6 matrices that behave like S_3.

More Notes (Class Summary)

6° $G = (R, F \mid R^3 = I,\ F^2 = I,\ RF = FR^2)$

You can see, by trying it out, that these rules give a complete description of S_3.

$$S_3 \longleftrightarrow \{I, R, R^2, F, RF, R^2F\}.$$

<u>Note:</u>

$$\begin{aligned} & RF = FR^2 \\ \Rightarrow\ & RF^2 = FR^2F \\ \Rightarrow\ & R = FR^2F \\ \Rightarrow\ & FR = F^2R^2F \\ \Rightarrow\ & FR = R^2F. \end{aligned}$$

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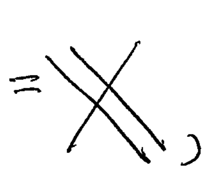
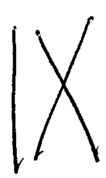
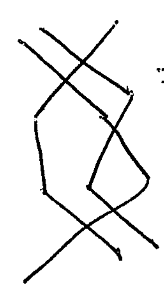
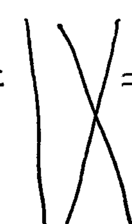
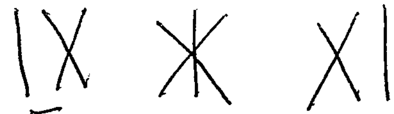
e.g.

$$\begin{aligned} (RF)R &= R(FR) \\ &= R(R^2F) \\ &= R^3F \\ &= F \checkmark \end{aligned}$$

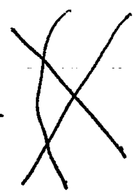
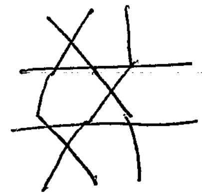
R = , F =  ⇒ RFR =  =  = F.

Sometimes, as above, one can give a succinct description of a group by <u>generators</u> (R, F) and <u>relations</u> ($R^3 = I$, ...).

7° 

$F_1 \quad F_2 \quad F_3$

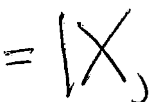
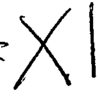
$S_3 \leftrightarrow (F_1, F_3 \mid F_1^2 = I,\ F_3^2 = I,\ F_1 F_3 F_1 = F_3 F_1 F_3)$

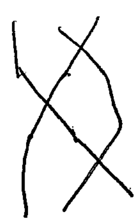
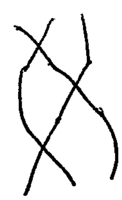
$F_2 =$  $=$  $= F_3 F_1 F_3.$

Let $G = (A, B \mid A^2 = B^2 = I,\ ABA = BAB)$

Then $G = \{I, A, B, AB, BA, ABA\}$

This gives us S_3 again.

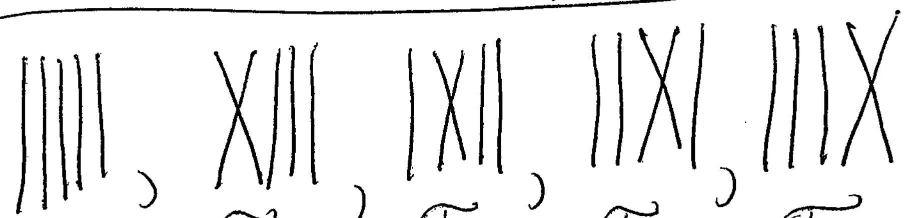
<u>Note</u>: ~~th~~ $A =$ , $B =$ 

$ABA =$  $=$  $= BAB.$

Work out the multiplication table again, using

(not to hand in)

8° S_5 + generalize



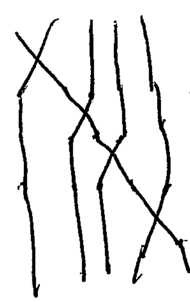
I, σ_1, σ_2, σ_3, σ_4

S_5 (with its $5! = 120$ elements) is generated by $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ + the relations

$$\boxed{\begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } |i-j| \geq 2. \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad i = 1,2,3 \end{array}}$$

Try working out some elements and products.

e.g. $\sigma_1 \sigma_2 \sigma_3 \sigma_4 =$

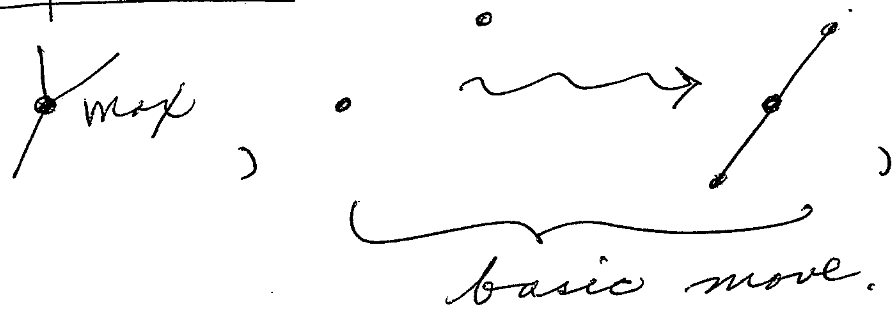


$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix} = (15432)$

Work out $(\sigma_1 \sigma_2 \sigma_3 \sigma_4)^2$.

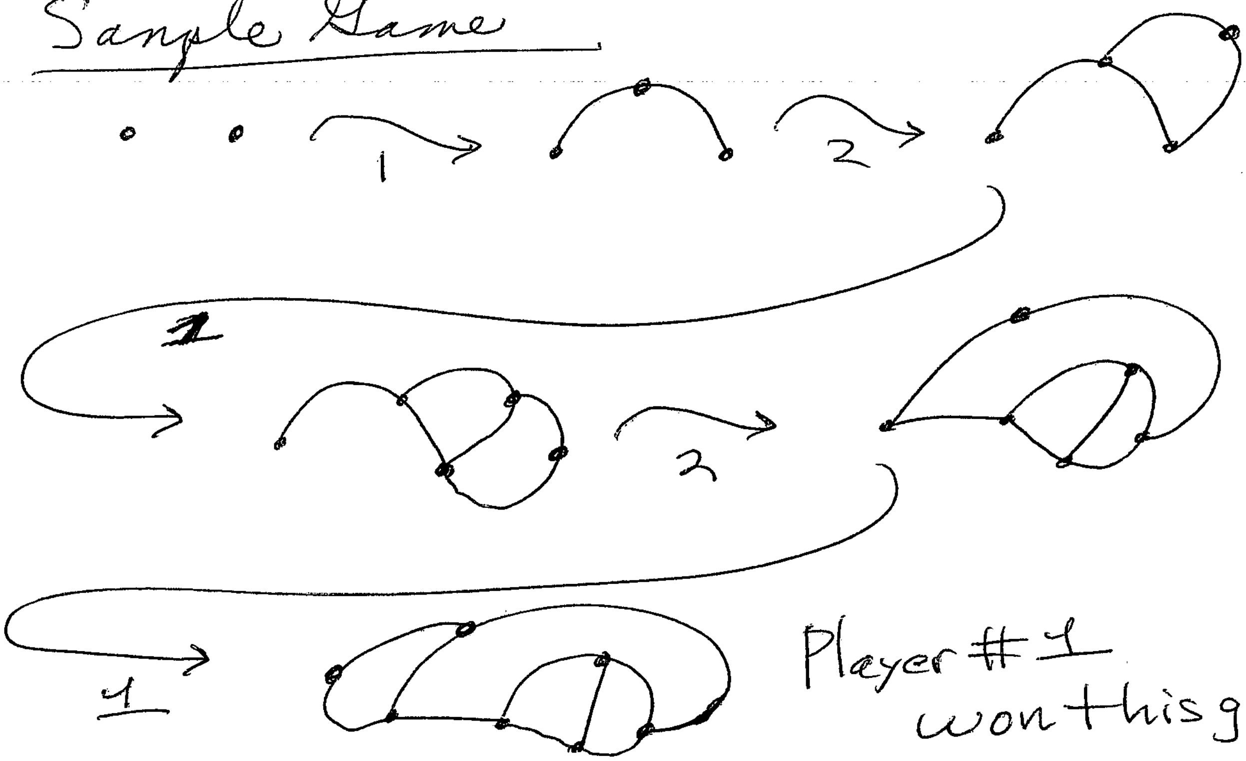
Project: Use above technique to investigate the subgroups of S_4. Note $\#(S_4) = 4! = 24$.

9. Sprouts



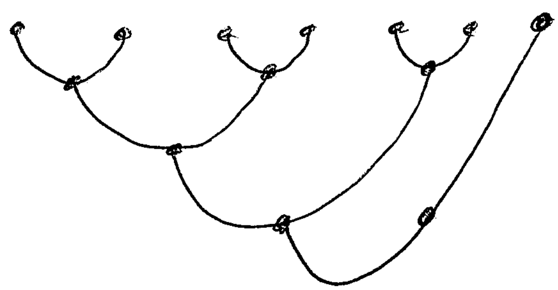
Goal: Be Last player to move.

Sample Game



Player #1 won this game.

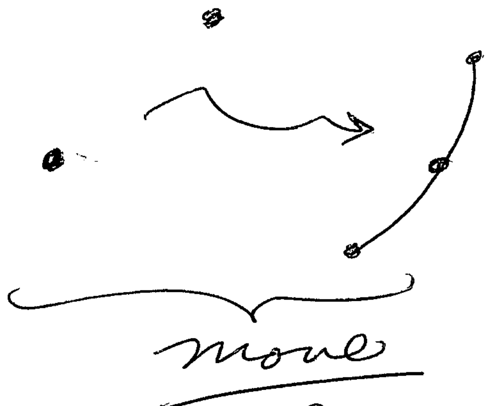
Example of some legal moves:



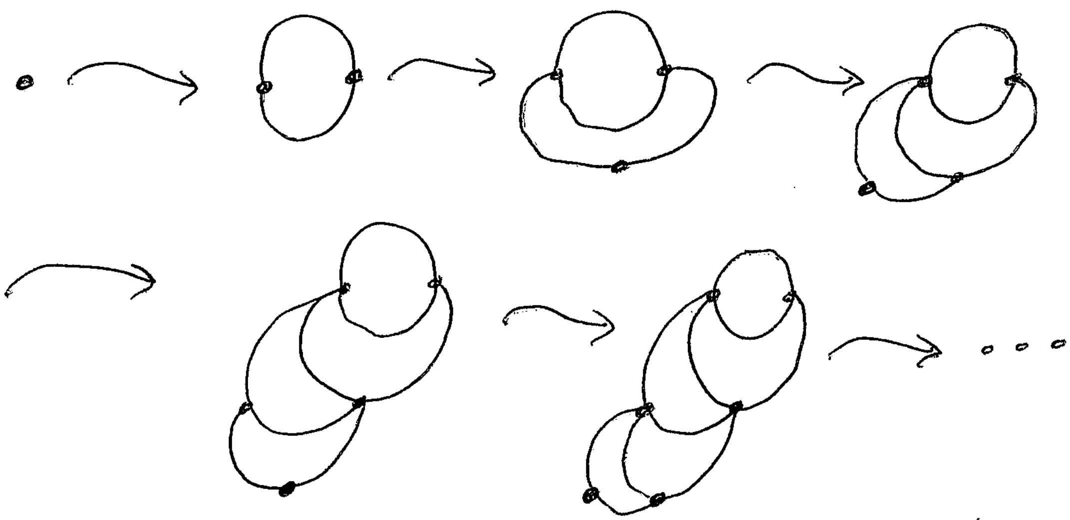
(related to our 2^n counting)

Theorem. Sprouts always ends in a finite number of moves. (Find an upper bound!)

10. BSprouts



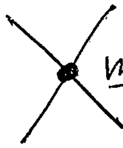
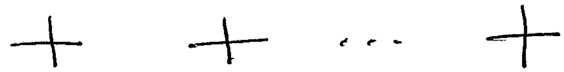
BSprouts can go on forever.



Problem. Make a game out of BSprouts.

The problem is to make some rule or rules that keep it from going on forever.

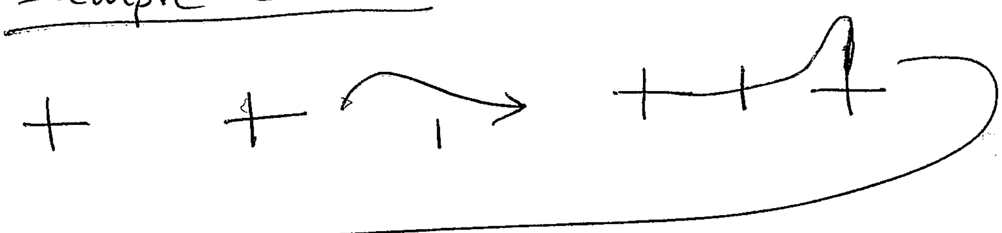
11° Brussels Sprouts

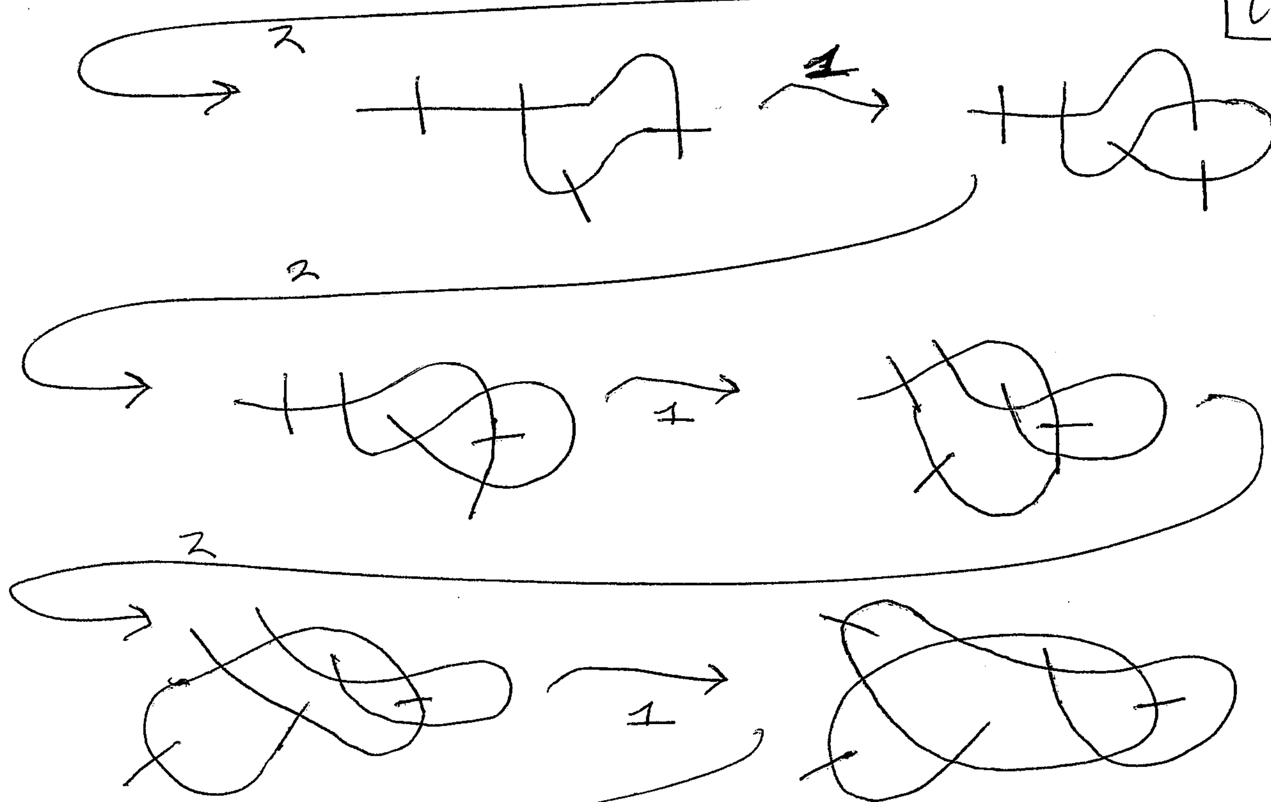
 max but given in this form at start. 

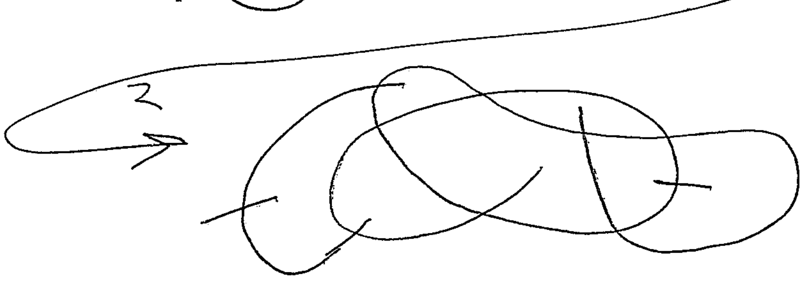
Winner makes last move.

Brussels Sprouts does always stop after a finite number of moves, but it has some quirks.

Sample Game







Win for 2nd player.