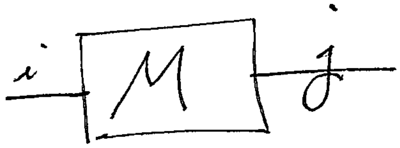
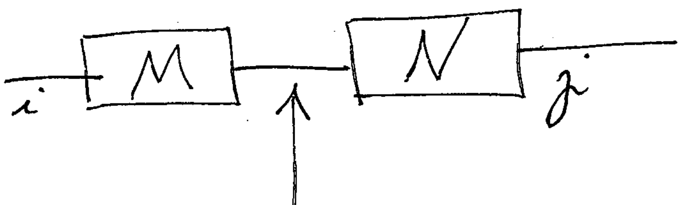
From Matrix Diagrams to the Associativity of Quaternion Multiplication

Matrix Mult

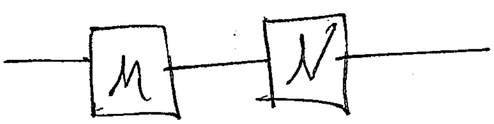
LK

$$\left.\begin{array}{l} M : M_{ij} \\ N : N_{kl} \end{array}\right\} \begin{array}{l} n \times n \\ \text{matrices} \end{array}$$

 $= M_{ij}$

 $\underset{\text{def}}{=} \sum_{k} M_{ik} N_{kj}$

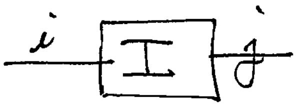
sum over all k on a closed edge

 $=$ 

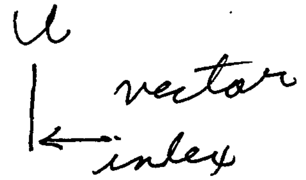
$$\boxed{(MN)_{ij} = \sum_{k} M_{ik} N_{kj}}$$

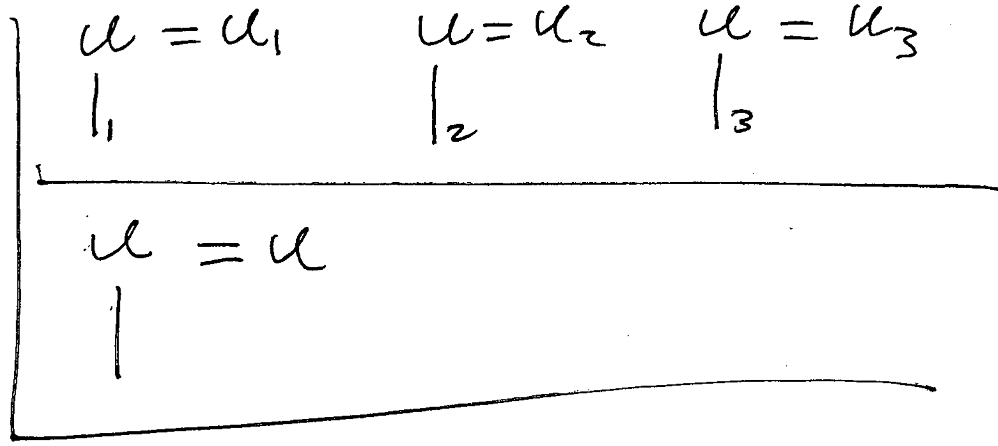
$$\| $$

$$(i^{th}\,\text{Row}(M)) \cdot (j^{th}\,\text{col}(N))$$

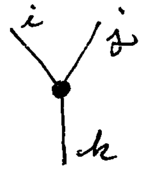
 $= I_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}$

Sometimes use $\equiv I$





$$\overline{\overline{\text{def}}} \sum_{i=1}^{3} u_i V_i = u \cdot V$$

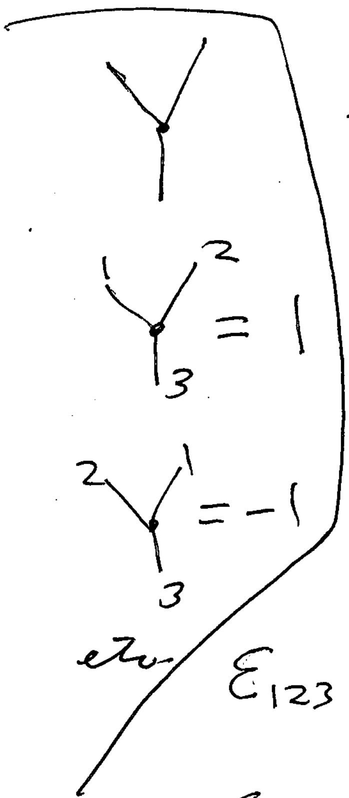


$\overline{\overline{\text{def}}}$ ε_{ijk} where

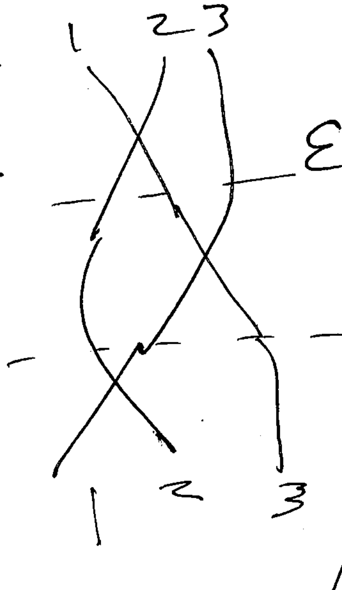
$\varepsilon_{123} = +1$
$\varepsilon_{132} = -1$
$\varepsilon_{213} = +1$
$\varepsilon_{231} = -1$
$\varepsilon_{312} = +1$
$\varepsilon_{321} = -1$

$\varepsilon_\sigma = \mathrm{sgn}(\sigma)$, $\sigma \in S_3$.

$\varepsilon_{ijk} = 0$ (if ijk not a perm of 123)



123 $\xrightarrow{-}$ 132 $\xrightarrow{-}$ 312



$\varepsilon_{112} = 0$ e.g.

$\varepsilon_{123} = 1$

$\varepsilon_{ijk} = -\varepsilon_{jik}$ etc

$\varepsilon_{112} = -\varepsilon_{112}$

$\therefore \varepsilon_{112} = 0$

$\varepsilon_{123\cdots n} = 1$

$\varepsilon_{i_1 i_2 \cdots i_n}$

$= -\varepsilon_{i_1 i_2 \cdots \underbrace{i_{k+1} i_k}\cdots i_n}$

A 3×3 *matrix*

$$|A| = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\, A_{1\sigma_1} A_{2\sigma_2} A_{3\sigma_3}$$

$\left(\sigma \equiv ijk \text{ all diff} \right)$

$\left(\sigma \equiv \text{"}\sigma_1\sigma_2\sigma_3\text{"} \right)$

+ 123
+ 231
+ 312

− 321
− 132
− 213

$$\left|\begin{array}{ccc} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{array}\right| \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{array}$$

− − − + + +

$$= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32}$$
$$- A_{13}A_{22}A_{31} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33}$$
$$= \underline{\underline{\sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\, A_{1\sigma_1} A_{2\sigma_2} A_{3\sigma_3}}}$$

123

321

$321 \rightleftarrows 312 \rightleftarrows 132$

$\downarrow$

123

$A_{2\times 2}$

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = \underset{12}{A_{11} A_{22}} - \underset{21}{A_{12} A_{21}}$$

$$= \sum_{\sigma \in S_2} \operatorname{sgn}(\sigma) A_{1\sigma_1} A_{2\sigma_2}$$

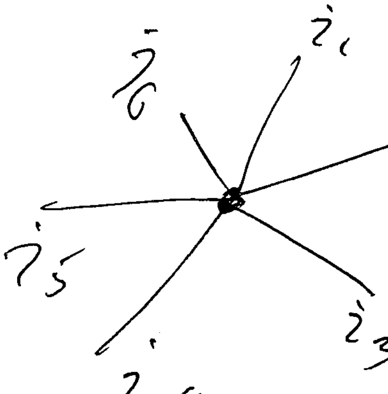
Genl def $A_{n\times n}$

$$|A| = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A_{1\sigma_1} \cdots A_{n\sigma_n}$$

Use to prove $|AB| = |A||B|$.

$$|A| = \sum_{i_1, \ldots, i_n = 1}^{n} \varepsilon_{i_1 \cdots i_n} A_{1 i_1} A_{2 i_2} \cdots A_{n i_n}$$

vanishes when $i_1 \cdots i_n$ not a perm.



$\varepsilon_{i_1 \cdots i_6}$

$k = 1, 2, 3$

$$\boxed{(u \times v)_k = \sum_{i,j=1}^{3} \varepsilon_{ijk} u_i v_j}$$

$$(u \times v)_1 = \sum_{i,j=1}^{3} \varepsilon_{ij1} u_i v_j$$

$$= \varepsilon_{231} u_2 v_3 + \varepsilon_{321} u_3 v_2$$

$$= +u_2 v_3 - u_3 v_2$$

$$(u \times v)_2 = \sum_{i,j} \varepsilon_{ij2} u_i v_j$$

$$= \varepsilon_{132} u_1 v_3 + \varepsilon_{312} u_3 v_1$$

$$= -u_1 v_3 + u_3 v_1$$

$$(u \times v)_3 = \sum_{i,j} \varepsilon_{ij3} u_i v_j$$

$$= \varepsilon_{123} u_1 v_2 + \varepsilon_{213} u_2 v_1$$

$$= u_1 v_2 - u_2 v_1$$

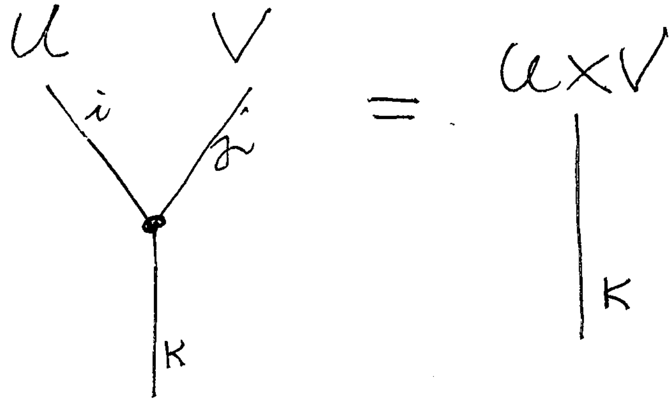
$$u \times v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\| $$

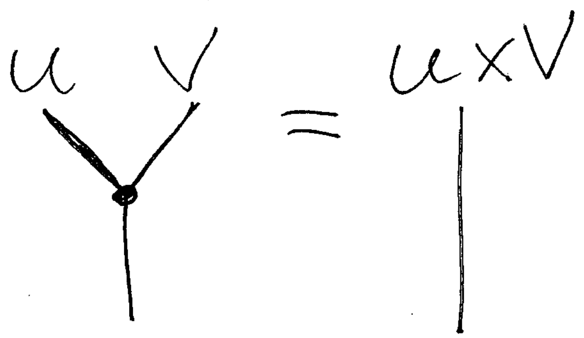
$$\vec{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \Big\} u_2 v_3 - u_3 v_2$$

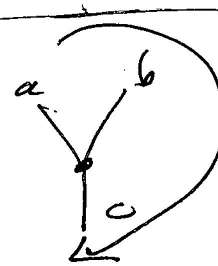
$$-\vec{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \Big\} \begin{matrix} -u_1 v_3 \\ +u_3 v_1 \end{matrix}$$

$$+\vec{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \Big\} \begin{matrix} u_1 v_2 \\ -u_2 v_1 \end{matrix}$$



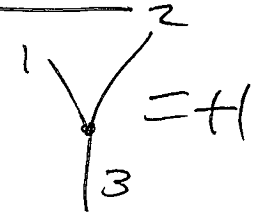
$$\boxed{(u\times V)_k = \sum_{i,j} \epsilon_{ijk} u_i V_j}$$

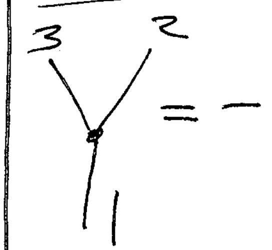




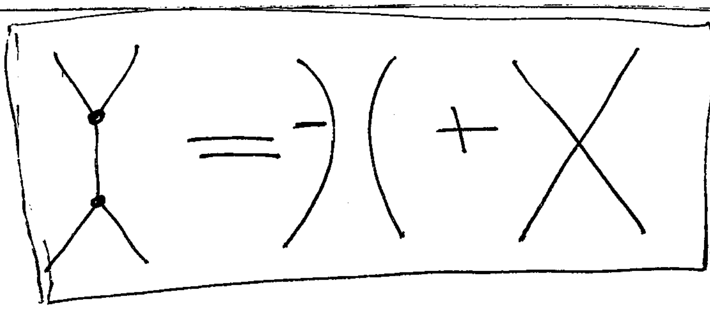
$= \epsilon_{abc}$

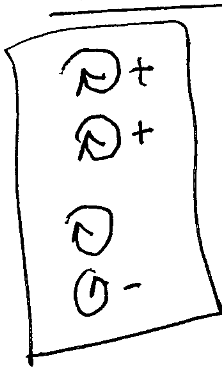
$$\epsilon_{abc} = \epsilon_{bca} = \epsilon_{cab}$$

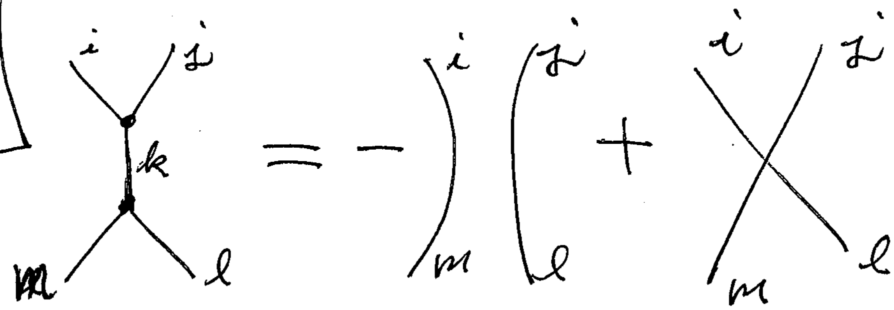




Lemma.







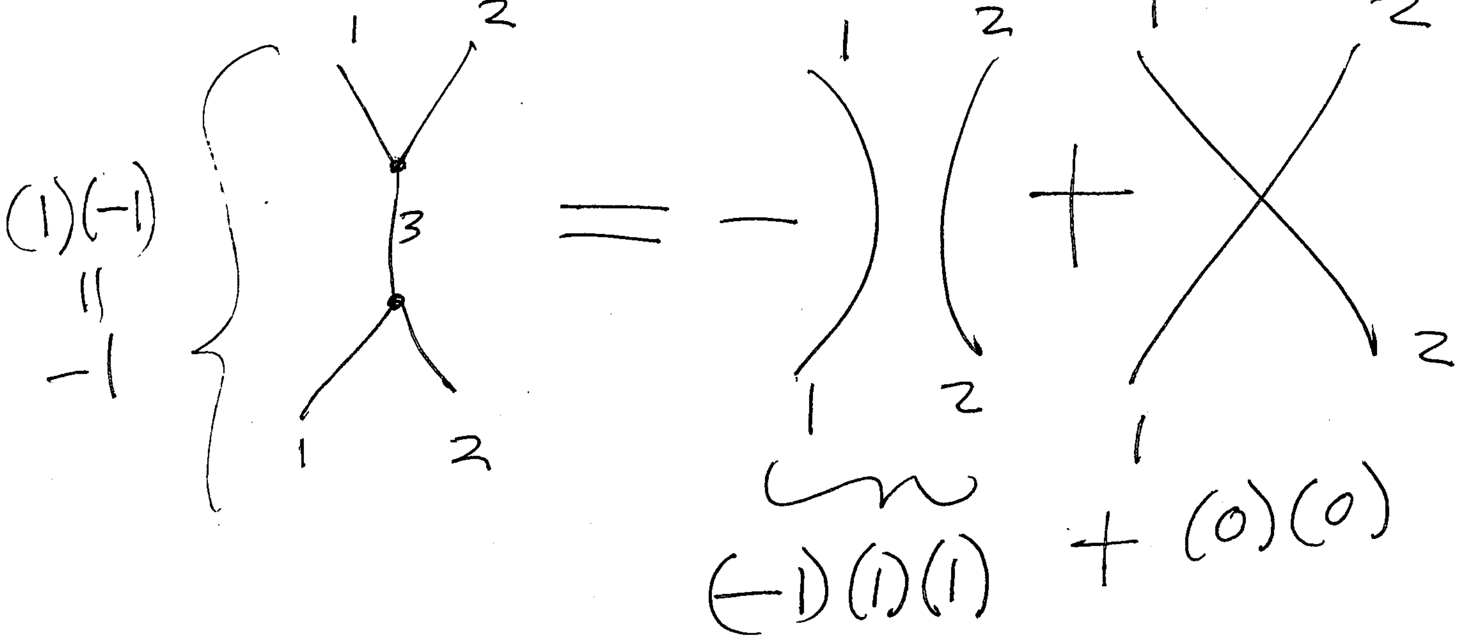
$$\sum_K \epsilon_{ijk}\epsilon_{klm} = -\delta_{im}\delta_{jl} + \delta^i_l \delta^j_m$$

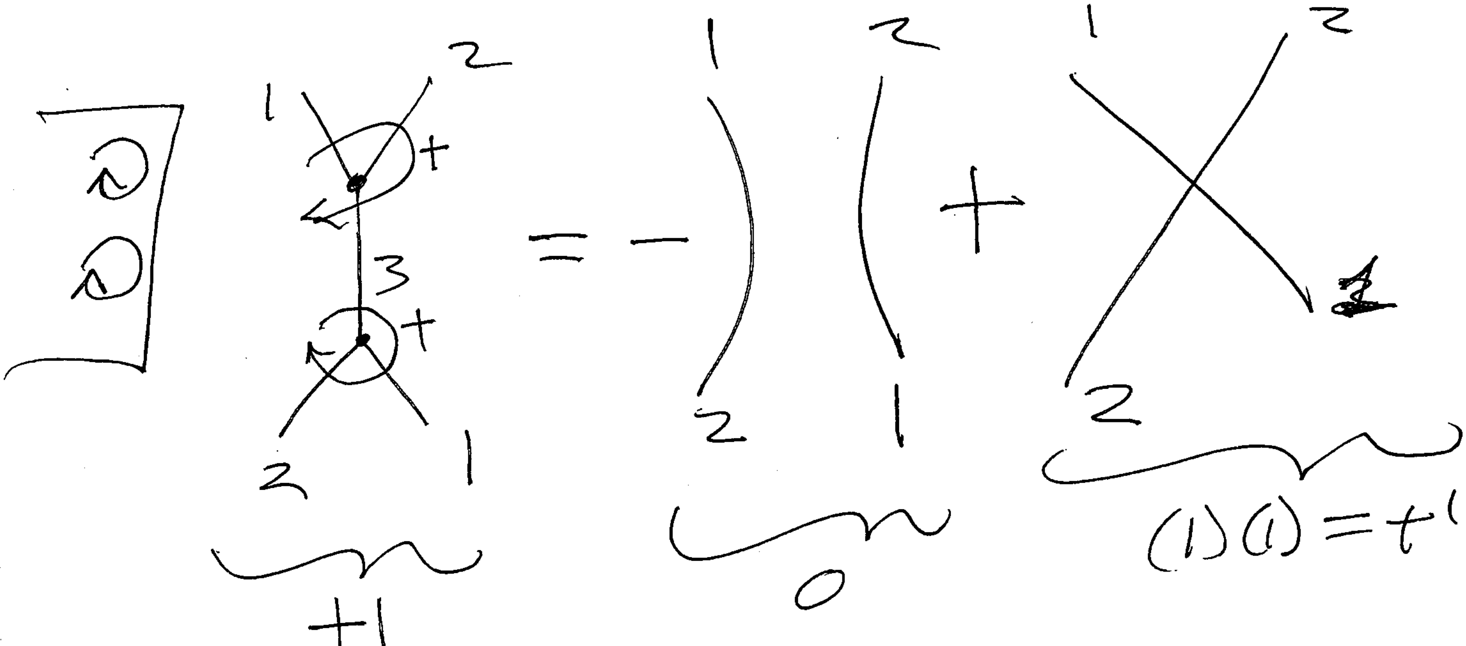
$= \delta_{ij}$

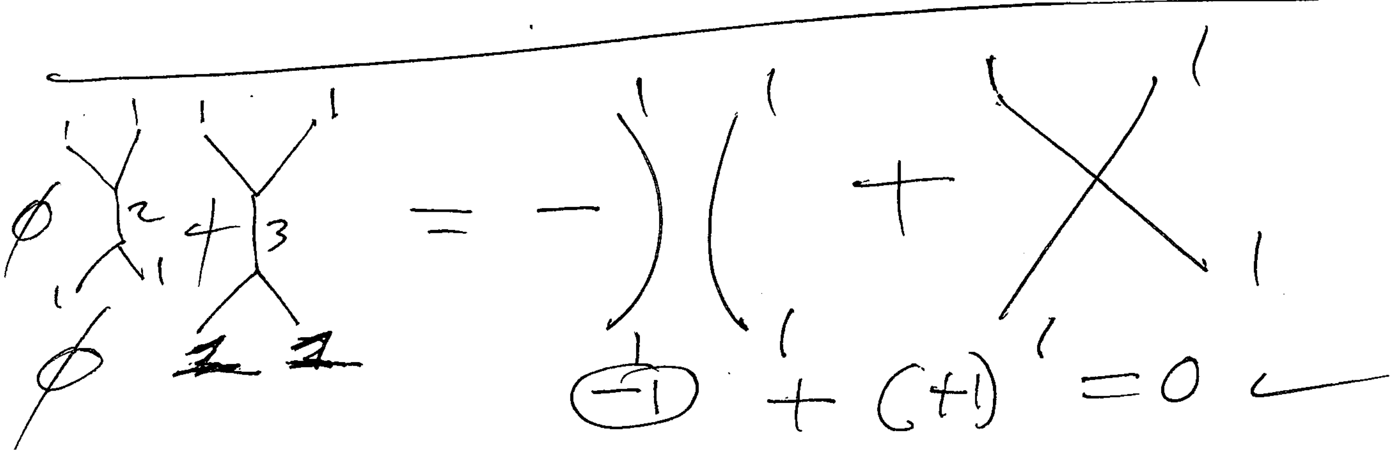
$= \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

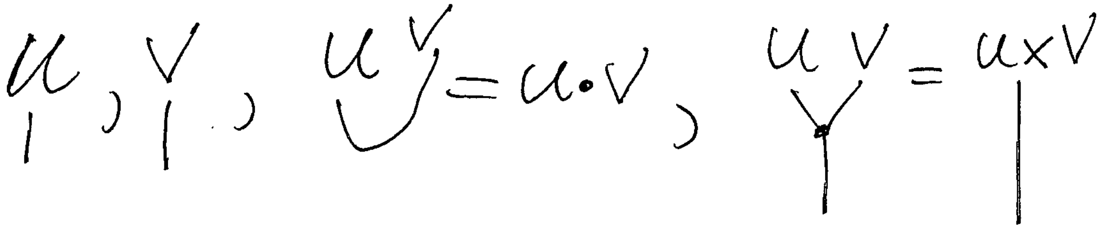
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta$

$\boxed{|^7_3 = 0,\ |^7_7 = 1 \text{ etc.}}$

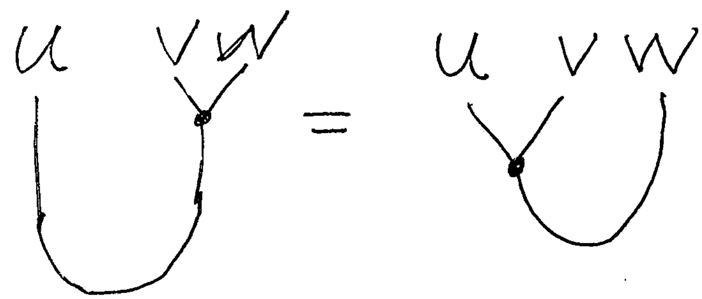


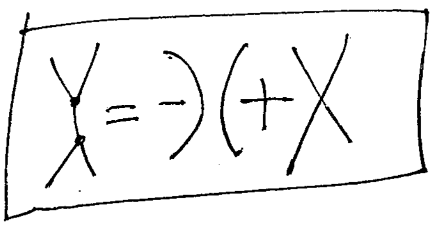






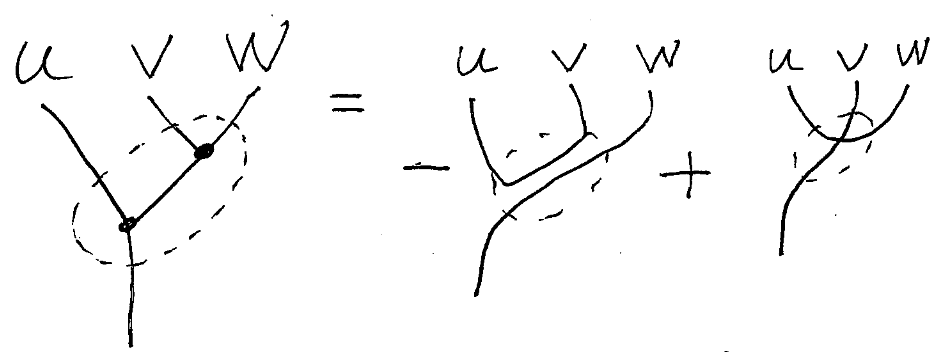
$u\cdot(v\times w) =$





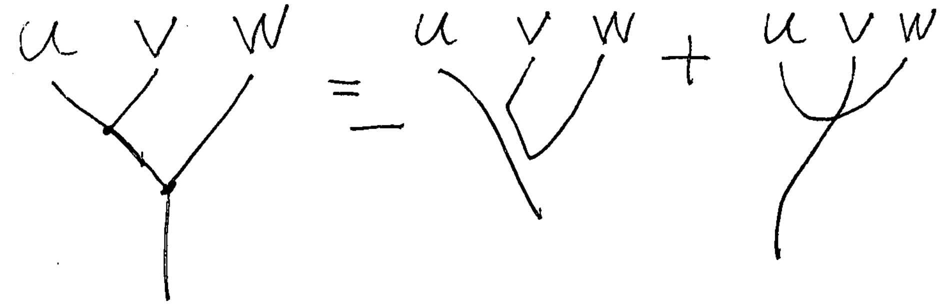
$= (u\times v)\cdot w$

$u\times(v\times w) =$



$= -(u\cdot v)w + (u\cdot w)v$

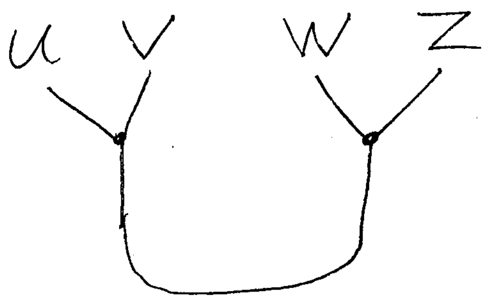
$(u\times v)\times w =$



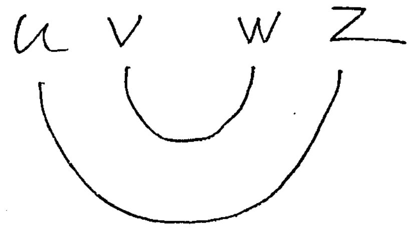
$= -(v\cdot w)u + (u\cdot w)v$

$$(u \times v) \cdot (w \times z)$$

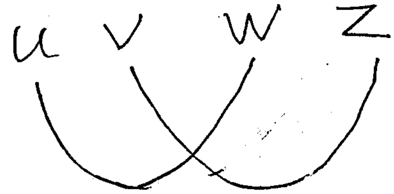
$=$



$= -$



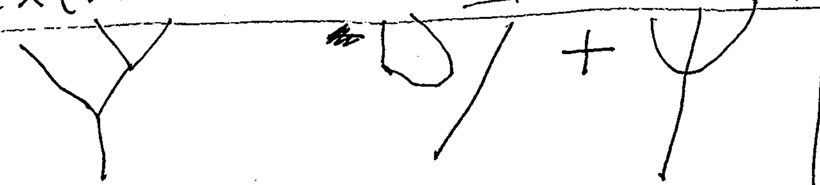
$+$



$$= -(u \cdot z)(v \cdot w) + (u \cdot w)(v \cdot z)$$

$$= \begin{vmatrix} u \cdot w & u \cdot z \\ v \cdot w & v \cdot z \end{vmatrix}$$

$u\times(v\times w) = -(u\cdot v)\,w + (u\cdot w)\,v$



$(u\times v)\times w = -(v\cdot w)\,u + (u\cdot w)\,v$

$(u\times v)\times w$

$u\cdot(v\times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$u(vw) = u(-v\cdot w + v\times w)$

$= -(v\cdot w)u + u(v\times w)$

$$\boxed{u(vw) = -(v\cdot w)u - u\cdot(v\times w) + \underline{u\times(v\times w)}}$$

$(uv)w = (-u\cdot v + u\times v)w$

$= -(u\cdot v)w + (u\times v)w$

$$\boxed{(uv)w = -(u\cdot v)w - (u\times v)\cdot w + \underline{(u\times v)\times w}}$$

$u(3+w) = 3u + uw$

$u \in \mathbb{R}^3$, 3 scalar, $w \in \mathbb{R}^3$

$= 3u - u\cdot w + u\times w$

etc.

$(s+v)(t+w)$

$= \underline{st} + \underline{sw} + tv + (-\underline{v\cdot w} + v\times w)$

$(st - v\cdot w) + (sw + tv - v\times w)$

Note: $u\cdot(v\times w) = (u\times v)\cdot w$

$\therefore$

$$u(vw) - (uv)w$$

$$= -(v\cdot w)u - u\cdot(v\times w) + u\times(v\times w) + (u\cdot v)w + (u\times v)\cdot w - (u\times v)\times w$$

$$= u\times(v\times w) - (u\times v)\times w + (u\cdot v)w - (v\cdot w)u.$$

$$= \cancel{-(u\cdot v)w} \cancel{+(u\cdot w)v} \cancel{+(v\cdot w)u} \cancel{-(u\cdot w)v} \cancel{+(u\cdot v)w} \cancel{-(v\cdot w)u}$$

$$= 0.$$

Thus quaternion mult is associative.