COMPLEX NUMBERS AND ALGEBRAIC LOGIC

by

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Abstract

The complex numbers (ii = -1) and the dual numbers (i*i = +1) are reconstructed from a point of view that is closely related to algebraic logic. A theorem is proved about the uniqueness of the multiplicative structure of these systems. The metric of Minkowski spacetime and a parametrization of the forward light-cone are shown to fit easily into the formalism given here.

1. Introduction

In [2] I introduced a construction that produces DeMorgan algebras from Boolean algebras, and pointed out its similarity to the construction of the complex numbers from the real numbers. In this paper I shall pursue this analogy further, producing what I believe is a useful new way to view the complex numbers.

First recall the method described in [2] for constructing a DeMorgan algebra from a given Boolean algebra B (with complementation $a \to a'$ and binary operations ab and $a+b$ satisfying the usual axioms). Let $T: B \to B$ be given by the formula $T(b) = b'$. Then T has no fixed point in the algebra B. In order to solve the equation $T(X) = X'$ and have the solution live in an appropriate algebra it is necessary to go to a larger domain $\hat{B}$. In [2] I let $\hat{B} = B \times B$ with new inversion $(a,b)' = (b',a')$, and binary operations $(a,b)(c,d) = (ac,bd)$, $(a,b) + (c,d) = (a+c,b+d)$. Then $\hat{B}$ is a DeMorgan algebra and there are many solutions to $X = X'$ in B. The two basic solutions are $\hat{\imath} = (1,0)$ and $\hat{\jmath} = (0,1)$.

More generally, suppose that $T : B \to B$ is given by the formula $T(x) = ax + bx'$ for some fixed elements $a,b \in B$. Then the sequence $T(x)$, $T^2(x)$, $T^3(x)$,... has period two. If we define $\hat{T} : \hat{B} \to \hat{B}$ by $\hat{T}(X) = aX + bX'$ $(a(b,c) = (ab,ac))$ then $(T(x), T^2(x))$ and $(T^2(x), T(x))$ are fixed points for $\hat{T}$. Thus fixed points for $\hat{T}$ are built from the sequence of iterates of T.

Now transpose these ideas over to the real numbers $\mathbb{R}$. Let $T : \mathbb{R}^\bullet \to \mathbb{R}^\bullet$ $(\mathbb{R}^\bullet = \mathbb{R} - \{0\})$ be given by the formula $T(x) = -x^{-1}$. Then $T(x) = x$ if and only if $x^2 = -1$. Iteration of T with starting value 1 produces the sequence 1, -1, 1, -1, 1,... . The analogy with DeMorgan algebras suggests that the two square roots of -1 could be represented by the ordered pairs [1,-1] and [-1,1].

In Section 2 I shall carry out an explicit reconstruction of the complex numbers that allows the identification $\sqrt{-1} = [1,-1]$. In the order of construction there is first an appearance of the dual numbers $a + bi$ where $i*i = +1$ ($*$ denotes the multiplication in the dual numbers). Let $\mathbb{C}$ denote the complex numbers and $\mathbb{D}$ the dual numbers. Let complex multiplication be denoted $\alpha\beta$ and dual multiplication be denoted $\alpha*\beta$. Then each multiplication can be expressed in terms of the other by the formulas.

$$\alpha\beta = \frac{1}{2}(\alpha*\beta + \alpha*\bar{\beta} + \bar{\alpha}*\beta - \bar{\alpha}*\bar{\beta})$$

$$\alpha*\beta = \frac{1}{2}(\alpha\beta + \alpha\bar{\beta} + \bar{\alpha}\beta - \bar{\alpha}\bar{\beta})$$

($\bar{\alpha}$ is the conjugate of α).

We shall show that the assumption of the existence of such a reciprocal relationship determines these formulas (Theorem 2.2).

We then obtain a very symmetrical view of the metric on Minkowski spacetime [5]. Let $\mathcal{M} = \mathbb{C} \times \mathbb{D}$ and $I : \mathcal{M} \to \mathbb{R}$ be given by $I(\alpha,\beta) = \alpha\bar{\alpha} - \beta*\bar{\beta}$. Then I corresponds directly to the usual spacetime interval. This is discussed in section 3. Section 4 discusses a parametrization of the forward light-cone in this formalism and its relationship with the Pauli spinor algebra. Section 5 compares our constructions with G. Spencer-Brown's primary arithmetic P and its natural dual P*.

At least on the level of analogy, complex numbers, dual numbers and spinors may all be regarded as outgrowths or extensions of Boolean algebra, hence as forms of algebraic logic.

2. Complex Numbers

The theme of this section is that complex numbers arise naturally in considering a process (or pattern) that may be viewed in two (at least) different ways. Perceptual examples of this phenomenon abound (figure-ground relationships, multiple interpretations of a scene, a word or a phrase, opposites that are seen to be parts of a larger whole). Perhaps the simplest example is an unending linear pattern of period two:

...ABABABABABABABABABABABABA...

This pattern can be seen either as an unending repetition of AB <u>or</u> as an unending repetition of BA. Each view reproduces the pattern, but structures it differently.

Let [A,B] denote the **AB** view and [B,A] denote the BA view. Define $\overline{[A,B]} = [B,A]$ to be the <u>conjugate</u> of [A,B].

A and B could be merely symbols on the page, or they could stand for entities with other structure. To obtain the complex numbers we assume that A and B represent real numbers and let $\mathbb{C} = \{[A,B] \mid A,B \in \mathbb{R}\}$. The following combination rules lend themselves for consideration:

$$\left.\begin{array}{ll} 1) & [A,B] + [C,D] = [A+C,B+D] \\ 2) & C[A,B] = [CA,CB] \\ 3) & [A,B]*[C,D] = [AC,BD] \end{array}\right\} A,B,C,D \in \mathbb{R}$$

Let $1 = [1,1]$ and $i = [1,-1]$ so that $i*i = +1$.

Note that $[A,B] = \left(\frac{A+B}{2}\right) 1 + \left(\frac{A-B}{2}\right) i$, and hence [A,B] has the form $a+bi$. Thus $\mathbb{C}$ with the $*$ multiplication is isomorphic to the dual numbers $\mathbb{D}$. (see also [3]).

Here $\mathbb{D} = \{a+bi \mid a,b \in \mathbb{R}\}$ and $\overline{a+bi} = a-bi$, $(a+bi)*(c+di) = (ac+bd)+(ad+bc)i$. The isomorphism $f\colon (\mathbb{C},*) \to \mathbb{D}$ is given by the formula $f([A,B]) = \frac{1}{2}(A+B) + \frac{1}{2}(A-B)i$. Note that $f(\overline{[A,B]}) = f([B,A]) = \overline{f([A,B])}$. Hence conjugation is preserved by f.

A little exploration now reveals that if we define a new multiplication on $\mathbb{C}$ with the formula $\alpha\beta = \frac{1}{2}(\alpha*\beta+\alpha*\bar{\beta}+\bar{\alpha}*\beta-\bar{\alpha}*\bar{\beta})$ for $\alpha,\beta \in \mathbb{C}$, then the resulting system is isomorphic to the usual complex numbers. This may seem to be a rather complicated route to $i^2 = -1$, but it is the point at which the snake bites its tail: As the next lemma shows, each multiplication is expressed in terms of the other by the same formula!

<u>Lemma 2.1</u>. Let $\alpha*\beta$ and $\alpha\beta$ be defined as above. Then $\alpha*\beta = \frac{1}{2}(\alpha\beta+\alpha\bar{\beta}+\bar{\alpha}\beta-\bar{\alpha}\bar{\beta})$.

The proof is a straightforward calculation and will be omitted. The specific form of this relationship between the two forms of multiplication is determined by symmetry assumptions:

Theorem 2.2. Let $\mathbb{C}$ be endowed with two multiplications $\alpha\beta$ and $\alpha*\beta$ satisfying:

i) $\alpha\beta = E_1\alpha*\beta + E_2\alpha*\bar{\beta} + E_3\bar{\alpha}*\beta + E_4\bar{\alpha}*\bar{\beta}$

$\alpha*\beta = E_1\alpha\beta + E_2\alpha\bar{\beta} + E_3\bar{\alpha}\beta + E_4\bar{\alpha}\bar{\beta}$ where

$E_1, E_2, E_3, E_4 \in \mathbb{R}$.

ii) $\alpha\beta$, $\alpha\bar{\beta}$, $\bar{\alpha}\beta$, $\bar{\alpha}\bar{\beta}$ are linearly independent as elements of the real vector space of functions from $\mathbb{C}\times\mathbb{C}$ to $\mathbb{C}$

iii) $\overline{\alpha*\beta} = \bar{\alpha}*\bar{\beta}$, $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$.

iv) $\alpha*\beta = \beta*\alpha$, $\alpha\beta = \beta\alpha$, $1*1 = 11 = 1$, $i*i = -ii$.

Then $(E_1,E_2,E_3,E_4) = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2}\right)$ or $\left(\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$.

Proof. It follows from i) that

$$E_1\alpha*\beta = E_1^2\alpha\beta + E_1E_2\alpha\bar{\beta} + E_1E_3\bar{\alpha}\beta + E_1E_4\bar{\alpha}\bar{\beta}$$

$$E_2\alpha*\bar{\beta} = E_2E_1\alpha\bar{\beta} + E_2^2\alpha\beta + E_2E_3\bar{\alpha}\bar{\beta} + E_2E_4\bar{\alpha}\beta$$

$$E_3\bar{\alpha}*\beta = E_3E_1\bar{\alpha}\beta + E_3E_2\bar{\alpha}\bar{\beta} + E_3^2\alpha\beta + E_3E_4\alpha\bar{\beta}$$

$$E_4\bar{\alpha}*\bar{\beta} = E_4E_1\bar{\alpha}\bar{\beta} + E_4E_2\bar{\alpha}\beta + E_4E_3\alpha\bar{\beta} + E_4^2\alpha\beta.$$

Hence

$$\alpha\beta = (E_1^2+E_2^2+E_3^2+E_4^2)\alpha\beta + 2(E_1E_2+E_3E_4)\alpha\bar{\beta}$$
$$+ 2(E_1E_3+E_2E_4)\bar{\alpha}\beta + 2(E_1E_4+E_2E_3)\bar{\alpha}\bar{\beta}.$$

Therefore it follows from ii) that

$$\left.\begin{aligned} E_1^2+E_2^2+E_3^2+E_4^2 &= 1 \\ E_1E_2+E_3E_4 &= 0 \\ E_1E_3+E_2E_4 &= 0 \\ E_1E_4+E_2E_3 &= 0 \end{aligned}\right\} \quad (**)$$

It is easy to see from this list of conditions that either all four E_k are non-zero or only one of is non-zero. In the latter case the possible solutions for $\alpha*\beta$ are $\pm\alpha\beta$, $\pm\bar{\alpha}\beta$, $\pm\alpha\bar{\beta}$, $\pm\bar{\alpha}\bar{\beta}$. However, by assumption ii) the middle two choices are not commutative thereby violating iii). The outer two choices do not satisfy $1*1 = 1$ and $i*i = -ii$. Thus we may assume that all of E_1, E_2, E_3, E_4 are non-zero. Further analysis now shows that $E_1^2 = E_2^2 = E_3^2 = E_4^2$ and hence $4E_k^2 = 1$, $k = 1,2,3,4$. It also follows from (**) that only <u>one</u> of the E_k can be negative. If E_2 or E_3 is negative then the multiplication will not satisfy iii). Hence the only possibilities are

$E_1 = \frac{1}{2},\ E_2 = \frac{1}{2},\ E_3 = \frac{1}{2},\ E_4 = \frac{-1}{2}$ and

$E_1 = \frac{-1}{2},\ E_2 = \frac{1}{2},\ E_3 = \frac{1}{2},\ E_4 = \frac{1}{2}$.

This completes the proof of the theorem.

Remark. It is worth noting that the view of complex numbers presented here fits very well with the usual geometric interpretation.

$$a + bi = [a+b,\ a-b]$$

We see $a+bi$ as a periodic oscillation between $a + b$ and $a - b$. Consider, in the complex plane, a point orbiting a on the real axis by a distance $|b|$. (See Figure 1.) The orbit intersects the real axis at $a + b$ and $a - b$.

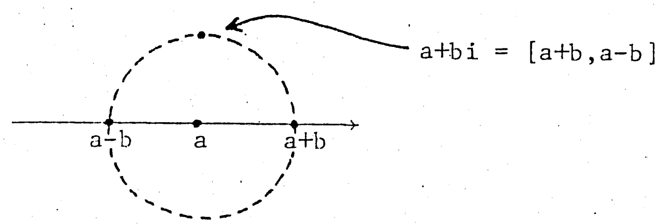


Figure 1

Remark. Let $\alpha = a+bi$. Then $\alpha\bar{\alpha} = a^2+b^2$ and $\alpha*\bar{\alpha} = a^2-b^2$. In the sequence **notation** we have $[A,B]\overline{[A,B]} = \frac{A^2+B^2}{2}$ and $[A,B]*\overline{[A,B]} = AB$. Define norms in the complex and dual numbers respectively by $N(\alpha) = \alpha\bar{\alpha}$, $N^*(\alpha) = \alpha*\bar{\alpha}$.

It has been remarked [4] that the norm in the dual numbers gives it the structure of a spacetime plane (one space coordinate and one time coordinate). Note that the lightcone (i.e. the set of points with vanishing norm) is then given by the set of $[A,B]$ such that $AB = 0$. Hence the lightcone consists of the multiples of $p = [1,0]$ and $q = [0,1]$.

Note that $\bar{p} = q$, $\bar{q} = p$, $p + q = 1$, $p*p = p$, $q*q = q$, $p*q = 0$, $p = \frac{1+i}{2}$, $q = \frac{1-i}{2}$.

Thus if we had constructed $\mathbb{D}$ over the two element Boolean algebra, then the lightcone would have formed a Boolean algebra with conjugation corresponding to complementation. The Boolean pattern is reflected in the real construction.

3. Minkowski Space

Let $\mathbb{C}$ denote the complex numbers and $\mathbb{D}$ denote the dual numbers. Define $I : \mathbb{C}\times\mathbb{D} \to \mathbb{R}$ by

$$I(\alpha,\beta) = \alpha\bar{\alpha} - \beta^*\bar{\beta} = N(\alpha) - N^*(\beta).$$

Lemma 3.1. If $\alpha = X + iY$, $\beta = T + iZ$ belong to $\mathbb{C}$ and $\mathbb{D}$ respectively, then

$$I(\alpha,\beta) = X^2 + Y^2 + Z^2 - T^2 .$$

Hence $\mathcal{M} = \mathbb{C}\times\mathbb{D}$ with the interval $I : \mathcal{M} \to \mathbb{R}$ can be identified with the Minkowski spacetime of special relativity [5].

Remark. This version of spacetime has appeared in a somewhat different guise as hermitian 2x2 matrices (see [6]). Let

$$\mathcal{H} = \left\{ \begin{pmatrix} T+Z & X-iY \\ X+iY & T-Z \end{pmatrix} \middle| \; X, Y, Z, T \in \mathbb{R} \right\}$$

Then

$$\text{Det} \begin{pmatrix} T+Z & X-iY \\ X+iY & T-Z \end{pmatrix} = T^2 - X^2 - Y^2 - Z^2.$$

This representation is formally identical to ours, but avoids the use of the dual numbers. Each representation has its own advantages.

Our representation, with its close symmetry between $\mathbb{C}$ and $\mathbb{D}$, is very suggestive mathematically. Here events of spacetime are paired points (α,β) from the mirrored planes $\mathbb{C}$ and $\mathbb{D}$. A point (α,β) is on the lightcone in four dimensional spacetime exactly when the norm of α in $\mathbb{C}$ agrees with the norm of β in $\mathbb{D}$. Each of $\mathbb{C}$ and $\mathbb{D}$ arise from the less structured $\mathcal{C}$, and $\mathcal{C}$ itself comes about by delineating views of the pattern ...ABABABAB... . In this sense spacetime arises naturally with the making of a distinction (compare [7]).

Remarkably, the hermitian viewpoint also has Minkowski space bound to a distinction. Here the arena is quantum mechanical (see [6]). Let S be a quantum system with two observable states P and Q. Then the wave-functions for S consist of all pairs of complex numbers z_1, z_2 such that

$|z_1|^2 + |z_2|^2 = 1$; the observables for the system correspond to the collection of hermitian sperators on S. Thus in this system the observables correspond to the space of 2x2 hermitian matrices $\mathcal{H}$.

It would be very interesting to better understand these relations between our mathematical view and the view point of quantum physics. (See also [0]).

I believe that the phenomenological view of complex numbers (as sketched in Section 2) is significant in this regard. Spacetime, for an observer, is constructed from an enormous number of distinctions generated just as we have generated the distinction between [A,B] and [B,A], or between $\mathbb{C}$ and $\mathbb{D}$ --- by framing a viewpoint of a groundform where a distinction is latent. To observe at all entails such choice. Mathematical spacetime is the bare bones of this process.

4. A Parametrization of the Light Cone

For each pair of complex numbers $(\alpha,\beta) \in \mathbb{C}\times\mathbb{C}$ form the following element Λ of $\mathcal{M} = \mathbb{C}\times\mathbb{D}$:

$$\Lambda = (\Lambda_1, \Lambda_2) = (\alpha\bar{\beta}, [\alpha\bar{\alpha}, \beta\bar{\beta}]) .$$

Lemma 4.1. The element Λ lies on the lightcone of $\mathcal{M}$. That is, $N(\Lambda_1) = N^*(\Lambda_2)$.

Proof. $N(\Lambda_1) = (\alpha\bar{\beta})\overline{(\alpha\bar{\beta})} = \alpha\bar{\alpha}\ \beta\bar{\beta} = N^*(\Lambda_2)$. This completes the proof.

Note that when $\alpha = 0$ or $\beta = 0$ then the expression for Λ reduces to the form $(0, [0,k])$ or $(0, [\ell,0])$, corresponding to the lightcone in the Minkowski plane as discussed at the end of Section 2. Since $\alpha\bar{\alpha}$ and $\beta\bar{\beta}$ are both positive, this does not parametrize the entire lightcone.

In fact, this parametrization is equivalent to one that arises from the Pauli spin matrices in the hermitian representation. That is, if

$$\begin{pmatrix} T+Z & X-iY \\ X+iY & T-Z \end{pmatrix} = T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + X\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + Y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + Z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

let $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$$\sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

The matrices $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ are called the Pauli spin matrices. They satisfy the identities $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_0$, $\sigma_1\sigma_2 = i\sigma_3$, $\sigma_2\sigma_3 = i\sigma_1$, $\sigma_3\sigma_1 = i\sigma_2$, and hence form a version of the quaternions.

To obtain a point on the lightcone let $u = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and $\bar{u} = (\bar{\alpha},\bar{\beta})$. Define $X^k = \bar{u}\sigma_k u$ for $k = 0, 1, 2, 3$. Then it is easy to verify that

$(X^0)^2 = (X^1)^2 + (X^2)^2 + (X^3)^2$ and hence that the X^k yield a parametrization of the positive light-cone (see [8]). In fact, it is not hard to verify that this parametrization via the Pauli matrices translates directly to the element   of Lemma 4.1. It correpsonds to the Hermitian matrix

$$\begin{pmatrix} \alpha\bar{\alpha} & \alpha\bar{\beta} \\ \bar{\alpha}\beta & \beta\bar{\beta} \end{pmatrix} .$$

Hopefully, this is only the beginning of an interaction between these formulations.

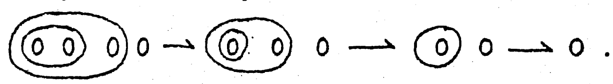
5. Primary Arithmetic and its Dual

In his book, Laws of Form [7], G. Spencer-Brown shows how to develop Boolean algebra from a ground that is simpler (at least to the eye of a geometer) than the usual set theoretic basis. Mathematically, the structure of this primary arithmetic is as follows: Let P denote all finite disjoint collections of circles (of various sizes) in the plane. Size does not really matter, and two such collections (called expressions) are regarded as the same if one can be re-arranged to the other by sliding or expanding the circles (but not pushing one circle through another). Thus

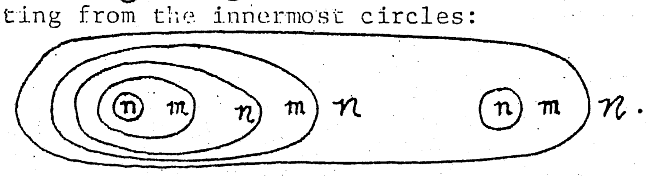
O O
O O   is the same expression as O O O O .

Two basic steps are introduced: O O ⇄ O (condensation) and ⓞ ⇄ (cancellation). In the second step the two concentric circles are removed from the configuration in which they stand. At the outset each type of step must be performed as specified. That is, condensation requires two empty adjacent circles, and cancellation requires two concentric circles with the inner circle empty.

Any expression can be uniquely reduced by these steps to a single circle, or eliminated entirely. For example:



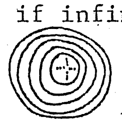
In order to see that the reduction is unique it suffices to give a well-defined method for computing this reduction directly from the expression. Let m denote the circle (marked) and n denote the blank state (unmarked). Then label the expression using ⓜ = n, ⓝ = m, mm = m, nn = n, mn = m, starting from the innermost circles:



This labelling process is easily seen to yield the same result as the step procedure. It amounts to viewing the expression as a signal-processing device with each circle in it acting as an inverter to the signals inside it.

This primary arithmetic P lends itself to algebraic description. For example ((a)) ⇄ a for any expression a in P. Note that in order to do this by a sequence of steps, we must first reduce a to m or n, then apply cancellation, then re-construct a. (⇄ can stand for a sequence of steps). Juxtaposition of expressions is the composition law in the algebra and is denoted by ab. Any number of expressions can be juxtaposed. Thus commutativity and associativity are implicit in the topological background.

The primary arithmetic contains no solution to (a) = a (corresponding to a′ = a) but there is an obvious solution if infinite expressions are allowed:

Let J =  be an infinite descend- array of concentric circles. Certainly J = (J). Adding such elements to P involves extensions analogous to the formation of DeMorgan algebras (see [3]).

Now consider that P itself is something like the complex numbers--its "numbers" live in the topological plane. Furthermore there is a candidate for "dual numbers" P*, obtained by switching the condensation and cancelation rules:

P	P*
O O ⇄ O	O O ⇄
(O) ⇄	(O) ⇄ O

P* is an "exterior" calculus; the result of a simplifying step procedure is obtained at once by a mod-2 count of the number of outer circles. That is, (a) ⇄ O in P* and so

(a) (b) ... (z) ⇄ O O ... O ⇄ O or

(blank) according as the number of circles is odd or even. P* is algebraically very weak (just so with $\mathbb{D}$) but it contains a solution to X ⇄ (X) (namely X = O).

Juxtaposition in P* can be expressed via the operations in P. The formula is

a*b = ((a) b) (a (b)). Then O*O = . Note the analogy with our formula (Lemma 2.1) expressing the relation between complex and dual multiplication.

I offer these remarks about P and P* in the hope that they will spark further development. The analogy with space-time is particularly strong: P represents the hierarchical time-bound, process-oriented, interior side. P* corresponds to the world of exterior surfaces of unknown internal structure. P and P*, taken together, yield a void: $O \underset{P}{\longrightarrow} OO \underset{P^*}{\longrightarrow}$.

In this regard it is worth noting that in our space-time interpretation of $\mathbb{C} \times \mathbb{D}$ the time coordinate occurs only in $\mathbb{D}$.

To quote an old result of George Gamow: The treasure is buried at $\sqrt{-1}$! (See [1] page 47).

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