An Introduction to Abstract Mathematics

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WAVELAND PRESS, INC.
Long Grove, Illinois

INTRODUCTION FOR THE STUDENT

This text and others like it are often described as *transition* books, primers for higher-level mathematics. What do we mean by a transition book and why is such a book necessary?

By now, you have seen a significant amount of mathematics, including at least a year or two of calculus and possibly some linear algebra. The mathematics in these courses is *quite* sophisticated. Calculus, for example, as developed by Newton and Leibniz, is the greatest mathematical achievement of the seventeenth century. The tremendous scientific advances of the last 300 years would not have been possible without the formulas and algorithms that follow from the theory of the integral and the derivative. Soon, you will take additional courses in such fields as probability, combinatorics, dynamical systems, linear programming, or topology, to list just a few examples. Given that calculus involves such high level mathematics, why does a math major need a *transition* course? Why not just plunge right into these so-called *advanced* courses?

One reason stems from the history of calculus itself. In the seventeenth and eighteenth centuries, mathematicians would manipulate infinite series much like ordinary finite sums. The results were usually quite correct, but the methods often led to errors. Here is an example:

The MacLaurin series expansion of $\ln(1 + x)$ is given by:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad (*)$$

This series converges for $-1 < x \leq 1$. Differentiating both sides of $(*)$ gives:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \qquad (**)$$

If we substitute $x = 1$ in (**), we get

$$\frac{1}{2} = 1 - 1 + 1 - 1 + \dots.$$

The right-hand side of the equation, however, is not a convergent series. What has gone wrong here is the indiscriminate differentiation of a power series, term by term, as if it were the same as a finite sum. Sometimes this can be done and sometimes it cannot. In fact, (**) is a true equation for all x such that $-1 < x < 1$. What becomes important is to *prove* under what conditions a power series *can* be differentiated term by term.

Another example is provided by letting $x = 1$ in (*) above, giving the true equation:

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots. \qquad (***)$$

Now if we rearrange the terms of the infinite series on the right-hand side, we obtain the equation:

$$\begin{aligned}\ln(2) &= \left(1 + \frac{1}{3} + \frac{1}{5} + \dots\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} \dots\right)\\ &= \left(1 + \frac{1}{3} + \frac{1}{5} + \dots\right) + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} \dots\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} \dots\right)\\ &= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots\right)\\ &= 0.\end{aligned}$$

Since we know that $\ln(2) \neq 0$, we have an apparent contradiction. The contradiction is resolved by noting that the right-hand side of (***) is a conditionally convergent series; that is, the series converges but if the terms of the series are replaced by their absolute values then the resulting series diverges. It can be proven that a rearrangement of a conditionally convergent series will not necessarily converge to the same sum as the original series. In fact, a conditionally convergent series can be rearranged to converge to any given number or even to diverge.

In both of these examples, mistakes are made by treating an infinite sum the same as a finite sum. In trying to determine which rules that apply to finite sums also apply to finite series, it is necessary first to *define* carefully what we mean by an infinite series and then *prove* properties of series using that definition. As each property is verified, it can be used to prove subsequent properties.

Mathematicians of earlier centuries commonly manipulated formulas and symbols indiscriminately without regard for whether or not those manipulations were justified. Nevertheless, often these "missteps" actually led to true formulas or provided insights into *why* something was true. The great mathematician **Leonhard Euler** (1707–1783) is famous for making discoveries in a totally nonrigorous way. Here is an example.

You may recall that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series by the so-called p-test. But knowing that the series converges does not tell you *to what number* the series converges. In fact, this series converges to $\frac{\pi^2}{6}$.

Euler's "proof" of this fact goes like this: the MacLaurin series expansion of $\sin x$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots.$$

Dividing by x gives the equation:

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots.$$

If we set $\frac{\sin x}{x} = 0$, the roots are: $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots.$

If we treat the infinite series as if it were a polynomial, as Euler did, then we can factor it as

$$\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\ldots$$

since this infinite product has the same roots and the same constant term as the infinite series.

So we get:

$$\begin{aligned}\frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots \\ &= \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\ldots \\ &= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\ldots.\end{aligned}$$

If we multiply out this last infinite product, as we would a finite product, we see that the coefficient of the x^2 term is the infinite series

$$-\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} - \ldots.$$

On the other hand, the x^2 term of the MacLaurin series is $-\frac{1}{3!} = -\frac{1}{6}$. Multiplying both expressions by $-\pi^2$ gives us Euler's result.

We emphasize that Euler was not indifferent to the idea of convergence of an infinite series and, of course, he knew that a power series was not the same as a polynomial. His insight and cleverness produced significant mathematics. He was later able to give a proof that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ that is considered rigorous by today's standards.

One of the important byproducts of finding a rigorous proof of a mathematical theorem is that it can often lead to new results or even to generalizations of the theorem, generalizations that may be impossible to discover by informal methods such as the ones employed by Euler.

For example, if you change the exponent in the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ from a 2 to another integer such as 3, 4 ... and so forth, you may well ask to what numbers these different series converge. There is a long history of attempts to answer this question.

First, we define a function $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ for s a real number greater than 1, so that $\zeta(2)$ is the series we considered above and $\zeta(2)=\frac{\pi^2}{6}$. (Note: the symbol ζ is the Greek letter zeta and the function $\zeta(s)$ is known as the Riemann zeta function, named after the mathematician **Bernhard Riemann** (1826–1866). You may recall the name Riemann from the study of Riemann sums in calculus.) Euler derived a formula for $\zeta(s)$ when s is an even positive integer. The formula involves a power of π and the so-called Bernoulli numbers, which we won't define here. No formula for $\zeta(s)$, when s is an odd positive integer, is known. In 1978, the French mathematician **R. Apery** proved that $\zeta(3)$ is irrational, but not much else is known about these numbers. This and countless other examples show that mathematics is not a closed subject. Many unsolved problems and even new areas of mathematics await the budding mathematician.

A transition book such as this one, then, is an introduction to the logic and rigor of mathematical thinking and is designed to prepare you for more advanced mathematical subjects.

We designed our course and this book with three goals in mind. First and foremost of these is to show you the elements of logical, mathematical argument, to have you understand exactly what mathematical rigor means and to appreciate its importance. You will learn the rules of logical inference, be exposed to definitions of concepts, be asked to read and understand proofs of theorems, and write your own proofs. At the same time, we want you to become familiar with both the grammar of mathematics *and its style*. We want you to be able to read and construct correct proofs, but also to appreciate different methods of proof (contradiction, induction), the value of a proof, and the *beauty* of an elegant argument.

A second goal is for you to learn how to do mathematics in a context, by studying real, interesting mathematics and not just concentrating on form. We have chosen topics that do not overlap significantly with other courses (such as the properties of the integers, the nature of infinite sets, and the complex numbers), that are essentially self-contained, and that will be useful to you later when you are exposed to more specialized, advanced mathematics.

Finally, we want you to realize that mathematics is an ongoing enterprise, with a long, fascinating, and sometimes surprising history. The notes sprinkled throughout the text are deliberately eclectic. "Historical Comments" give you pictures of the tremendous successes (and equally spectacular failures) of brilliant mathematicians of the past. "Mathematical Perspectives" may spotlight questions that are still unanswered and are the subject of current research, or that simply show an interesting further aspect of the material you have just studied.

This proves $P(n + 1)$ and by induction it follows that $P(n)$ is true for all positive integers n.

Corollary 5.2.8 $\sum_{k=0}^{n} \binom{n}{k} = 2^n.$

PROOF: Apply the Binomial Theorem with $a = b = 1$.

Note: The Binomial Theorem is also true when a and b are real numbers. The reason is that the axioms of addition and multiplication that hold in **Z** (Axioms 1–8) also hold in **R**. These matters are discussed in detail in Chapter 7.

MATHEMATICAL PERSPECTIVE: BERNOULLI NUMBERS

We conclude this section with a discussion of the Bernoulli numbers, which were alluded to in the Introduction. The Bernoulli numbers are a sequence of rational numbers defined by **Jakob Bernoulli** (1654–1705). They have had some very interesting applications in mathematics over the last several hundred years. As background, consider the following formulas:

$$1 + 2 + 3 + \ldots + (n-1) = \frac{n(n-1)}{2}$$

$$1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}$$

$$1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 = \frac{n^2(n-1)^2}{4}.$$

These formulas for sums of powers can be proven by induction. The first is an easy consequence of the formula in Example 1 and the other two are exercises.

Can you see any pattern to these formulas? Nothing is immediately obvious, but note that the right-hand side of each equation is a polynomial expression in the variable n. The first is $\frac{1}{2}n^2 - \frac{1}{2}n$, the second $\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$, and the third $\frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2$. Note that each of these polynomials has 0 as its constant term and that the highest power of n that appears is one more than the exponent on the left-hand side of the equation. In looking at other formulas of the form $1^k + 2^k + 3^k + \ldots + (n-1)^k$, Bernoulli noted that the sum was a polynomial in n of degree $k + 1$ and 0 constant term. The coefficient of n in these polynomials takes on the values $-\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0$ for $k = 1, 2, 3, 4, 5$. These are the first five Bernoulli numbers. Bernoulli was led to the following recursive definition.

Definition 5.2.9 The sequence of numbers $B_0, B_1, B_2, \ldots$, called **Bernoulli numbers**, are defined by $B_0 = 1$ and if $B_0, B_1, B_2, \ldots, B_{t-1}$ are defined then

$$B_t = -\frac{1}{t+1}\sum_{j=0}^{t-1}\binom{t+1}{j}B_j.$$

Using this formula for $t = 1, 2, 3, 4$, we obtain the following equations:

$$B_1 = -\frac{1}{2}\binom{2}{0}B_0 = -\frac{1}{2}$$

$$B_2 = -\frac{1}{3}\left[\binom{3}{0}B_0 + \binom{3}{1}B_1\right] = \frac{1}{6}$$

$$B_3 = -\frac{1}{4}\left[\binom{4}{0}B_0 + \binom{4}{1}B_1 + \binom{4}{2}B_2\right] = 0$$

$$B_4 = -\frac{1}{5}\left[\binom{5}{0}B_0 + \binom{5}{1}B_1 + \binom{5}{2}B_2 + \binom{5}{3}B_3\right] = -\frac{1}{30}.$$

Other Bernoulli numbers can be computed in similar fashion. It should be noted that if t is an odd positive integer >1, then $B_t = 0$. This fact is not easy to prove from our definition.

Using these numbers Bernoulli was able to give a formula for the sum of the first n kth powers. We state the result without proof. You are encouraged to learn the proof on your own. One source is *A Classical Introduction to Modern Number Theory* by K. Ireland and M. Rosen [10].

Theorem 5.2.10 **Bernoulli** If k is a positive integer, then

$$1^k + 2^k + 3^k + \ldots + (n-1)^k = \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j n^{k+1-j}.$$

The reader should verify the formula for $k = 1, 2, 3$ to see that it agrees with the formulas given.

Another application of the Bernoulli numbers is to the Riemann zeta function, which was discussed in the Introduction. Recall that for a real number $s > 1$, $\varsigma(s) = \sum_{n=1}^{\infty} 1/n^s$. Euler proved that $\varsigma(2) = \pi^2/6$ and in the Introduction we gave his informal proof of that fact. Euler generalized that result to even integers and again Bernoulli numbers are involved. More specifically, he proved that if k is a positive integer, then

$$\varsigma(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}.$$

As an example, we see that the series $\sum_{n=1}^{\infty} 1/n^4$ converges to $\pi^4/90$.

Exercises 5.2

1. Prove the following formulas using mathematical induction.
   (a) $1 + 3 + 5 + \ldots + (2n - 1) = n^2$.
   (b) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.
   (c) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

2. Prove the following:
   (a) $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \dfrac{(2n-1)(2n)(2n+1)}{6}$.
   (b) $2^2 + 4^2 + 6^2 + \ldots + (2n)^2 = \dfrac{(2n)(2n+1)(2n+2)}{6}$.

3. Prove that if a is any real number except 1, then

$$1 + a + a^2 + a^3 + \ldots + a^n = \frac{(a^{n+1} - 1)}{a - 1}.$$

4. (a) Prove that $2^n > n^2$ for all integers $n \geq 5$.
   (b) Prove that $2^n < n!$ for all $n \geq 4$.

5. Let $a, b_1, b_2, \ldots, b_n \in \mathbf{Z}$. Prove that $a(b_1 + b_2 + \ldots + b_n) = ab_1 + ab_2 + \ldots + ab_n$.

6. Let $f\colon \mathbf{Z}^+ \to \mathbf{Z}^+$ be defined recursively by $f(1) = 1$ and $f(n + 1) = f(n) + 2^n$ for all $n \in \mathbf{Z}^+$. Prove that $f(n) = 2^n - 1$.

7. Let $f\colon \mathbf{Z}^+ \to \mathbf{R}$ be defined recursively by $f(1) = 1$ and $f(n + 1) = \sqrt{2 + f(n)}$ for all $n \in \mathbf{Z}^+$. Prove that $f(n) < 2$ for all $n \in \mathbf{Z}^+$.

8. The **Fibonnaci numbers** f_n, $n = 1, 2, 3, \ldots$, are defined recursively by the formulas $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.
   (a) Write out the first ten Fibonnaci numbers.
   (b) Compute $f_1 + f_2, f_1 + f_2 + f_3, f_1 + f_2 + f_3 + f_4$.
   (c) Derive a formula for the sum of the first n Fibonnaci numbers and prove it by induction.
   (d) Prove that $f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1}$ for all $n \geq 1$.

9. Let H_n be the number of handshakes required if in a group of n people each person shakes with every other person exactly once.
   (a) Compute H_n for $n = 2, \ldots, 6$.
   (b) Find a recursion formula for H_{n+1} in terms of H_n.