

An Innocent Investigation

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January 2006

The beginning. Have you ever wondered why every number is either even or odd? I don't mean to ask if you ever wondered *whether* every number is either even or odd, but *why*. Hardly anyone except the most hardened skeptic would even question that every number is either even or odd. But you might have asked why. Then again, maybe the two questions aren't that different.

So, how do we know every number is either even or odd? Experience, for one thing. We know the numbers 2, 4, 6, 8, 10, 12, and so forth, are all even; and the numbers 1, 3, 5, 7, 9, 11, and so forth, are all odd. We see the pattern, namely odd, even, odd, even, odd, even, and so forth, and that's pretty convincing.

For another thing, we all learned that numbers are even or odd at a very young age. We learned it so early that it probably didn't even occur to us at the time to question it. That's a sort of knowledge by authority. But even then the authorities probably gave us some reason why it's true, and by now we have forgotten that reason.

What are we looking for? Where do we look to find out why all numbers are even or odd? How do we know when we've found the answer? Are there any guidelines?

Let's go blindly forward, and, maybe we'll find some answers.

What do we know? Maybe if we look at something simpler we know, and analyze that, we'll get somewhere. Let's start with the statement made above: We know the numbers 2, 4, 6, 8, 10, 12, and so forth, are all even; and the numbers 1, 3, 5, 7, 9, 11, and so forth, are all odd.

How do we know 2, 4, 6, 8, 10, 12, and so forth, are all even? What does it even mean that 2, 4, 6, 8, 10, 12, and so forth, are all even? There are a couple of answers that come to mind. One is that the number 2 evenly divides them. Another is that each of them is the sum of some other number taken twice. The second statement seems to depend on the concept of addition, while the first depends on the concept of division, so, maybe the second concept is simpler. Let's analyze it a little more.

For instance, we're saying a number, like 10, is even, because it is the sum of some other number, 5, taken twice, that is, 10 is even since $10 = 5 + 5$. That looks like a pretty good definition. We can say it in words, as we just did, or we can say it more formally, using symbolism, as follows. A number n is said to be *even* if there exists a number m such that $n = m + m$. The phrase "there exists a number n " is an example of an *existential quantifier*. There is a standard notation for existential quantifiers, and we can use it to abbreviate this definition a little bit.

Definition. A number n is *even* if

$$\exists m : n = m + m.$$

We read that defining clause as "there exists an m such that n equals m plus m ." Note that the colon for "such that" is only one abbreviation that has been used for "such that," but let's stick to it here for consistency.

With this definition, we can show that the numbers 2, 4, 6, 8, 10, and 12, are all even, since $2 = 1 + 1$, $4 = 2 + 2$, $6 = 3 + 3$, $8 = 4 + 4$, $10 = 5 + 5$, and $12 = 6 + 6$. The "and so forth" gives us a little

trouble since it's not exactly clear what how we're supposed to go forth and what happens when we do.

How about the other half of the statement, that part that says "the numbers 1, 3, 5, 7, 9, 11, and so forth, are all odd?" What is an odd number? It means when you try to divide the number into two equal parts, you can't because you'll have an odd 1 left over. Let's suppose that we try to formalize that a bit. Then n is odd when it is of the form $m + m + 1$. We could make that a definition. A number n is *odd* if $\exists m : n = m + m + 1$.

What do we count as a number? There's a little problem here. Certainly, 5 is odd since $5 = 2 + 2 + 1$, and 3 is odd since $3 = 1 + 1 + 1$. But what about 1? We want 1 to be odd, but $1 = 0 + 0 + 1$. Do we take 0 to be a number or not? If we don't, then we have to change the definition of odd so it accommodates 1. But if we do, then our investigation may become more complicated. And what of negative numbers? Do we include them? And fractions? Certainly not fractions, because a number like 1 becomes even since $1 = \frac{1}{2} + \frac{1}{2}$ and we'd have to change our definition of even.

So, when we talk about "number" here, we exclude fractions. We could include negative numbers, if we wanted to, since negative numbers are either even or odd. And we could include 0 even if we excluded negative numbers, if we wanted to. But trust me, it'll turn out easier if we exclude negative numbers. And I'd like to exclude 0, not because it's easier, but for other reasons. Here's one reason. When we first learned that numbers were either positive or negative, we probably didn't think of 0 as being a number yet; that probably came later in our education. The other reason is historical. Historically, when people first studied this question, 0 wasn't thought to be a number. Neither of these reasons are commanding, but, since I'll want to refer to early history later, it would be nice to only include positive numbers.

So, let's exclude 0 from our discussion of even and odd numbers. Then we have a problem. The number 1, which we want to be odd, is not of the

form $1 = m + m + 1$. How should we fix up our definition of odd so that 1 is included? One possibility is to say a number n is odd if it's of the form $m + m - 1$; another is to say n is odd if either $n = 1$ or n is of the form $m + m + 1$. The second one is clumsy. Let's use the first one.

What? Clumsy? What does that have to do with what we're doing? Not an awful lot, but when there's a choice between two things, and one is clumsier than the other, choose the other, that is, the more elegant choice. Not just because it's more elegant, but because it may lead to simpler analysis later on. We generally understand things better when they're described in simpler terms, so long as those terms remain accurate and complete.

So, it looks like we have a definition of odd numbers. We can make it a little bit simpler if we change add 1 to each side of the defining equation so that instead of $n = m + m - 1$ we have $n + 1 = m + m$. Subtraction is determined by addition, so it does make the definition more primitive if it's stated in terms of addition.

Definition. A number n is *odd* if

$$\exists m : n + 1 = m + m.$$

In other words, a number n is odd if the next number, $n + 1$, is even.

Along the way to this definition, we made the decision that our numbers are supposed to exclude fractions, negative numbers, and 0.

The pattern. We now know what we mean by even and odd. But before going on to the big job of showing every number is even or odd, let's look a bit the relation between even and odd numbers.

We saw a pattern for numbers earlier, namely odd, even, odd, even, odd, even, and so forth. How can we explain that pattern? We can restate it by saying odd numbers are followed by even numbers while even numbers are followed by odd numbers. Can we explain why those two statements are correct?

Let's start with the claim that an odd number is followed by an even number. But that's just the definition. In order for a number n to be odd, it is

required that the next number $n + 1$ be of the form $m + m$, that is, that $n + 1$ be even.

Now let's consider the claim that an even number is followed by an odd number. Let the even number be n , so that the next number is $n + 1$. Now, since n is even, we know there is a number m so that $n = m + m$. How are we going to conclude that the next number $n + 1$ is odd? In order to show that $n + 1$ is odd, according to our definition, we need to show that the number after it, $n + 2$ is even. But n was even, that is, $n = m + m$, therefore $n + 2 = m + m + 2 = (m + 1) + (m + 1)$, so $n + 2$ is even, as required. That's a good explanation.

Now that we've found explanations, let's write them up formally for the record. We'll do that by stating the claim as a theorem along with the proof we just found.

Theorem. The number after an odd number is even, and the number after an even number is odd.

Proof. The first statement is the definition for odd number.

Next, let n be an even number. Then $n = m + m$ for some number m . Therefore, $n + 2 = (m + 1) + (m + 1)$ is also an even number. Since the number after $n + 1$ is even, therefore, by the definition for odd number, $n + 1$ is odd. Q.E.D.

We now understand why the pattern goes odd, even, odd, even, odd, even, and so forth. We still have to figure out how to use that knowledge to show every number is either even or odd.

General principles. Here's one way that might help. To figure out the *parity* of n , that is, whether n is even or odd, look at its predecessor $n - 1$. If $n - 1$ is odd, then n is even, but if $n - 1$ is even, then n is odd. Thus if the predecessor $n - 1$ is either even or odd, then so will n be, but with the opposite parity.

That helps, but we still have two problems. (1) How do we know n has a predecessor? (2) How do we know the predecessor is either even or odd?

Let's look at (1) first. It may be that n *doesn't* have a predecessor. That happens when $n = 1$. But, of course, 1 is the smallest number, and doesn't have a predecessor. In this even/odd in-

vestigation, that's not a problem since we know 1 is odd. But we have identified two general principles, the first statement so basic that we'll call it an axiom and not try to prove it, and the second we'll prove after we find another axiom.

Axiom. 1 is a number that has no predecessor, that is, there is no number m such that $1 = m + 1$.

Theorem. Every number other than 1 does have a predecessor, that is, if $n \neq 1$, then $\exists m : n = m + 1$. (Proof to come later.)

These two statements take care of problem (1). But we still have problem (2) to address. There are a three of ways to go, all more or less equivalent.

Infinite descent. This will be a long trip. We want to know n is either even or odd. If we knew its predecessor $n - 1$ was either even or odd, we'd be done. If $n - 1$ were 1, we'd be done, but, if not, we'd need to know its predecessor $n - 2$ was either even or odd. If that were 1, we'd be done, but, if not, we'd need to know its predecessor was either even or odd. And so forth. If at any stage we reach 1, we would be done, but if we never reach 1, then we would have an infinite decreasing sequence of numbers $n, n - 1, n - 2, \dots$ that never ended. But, of course that can't be. We've found another general principle of numbers.

Axiom. There is no infinite decreasing sequence of numbers.

This is a somewhat unsatisfactory axiom because it explicitly mentions infinity. It would be nice to have a general principle that didn't depend on the concept of infinity.

Mathematical induction. A standard way to avoid this infinity is to appeal to a different general principle called mathematical induction. Although it's a different principle, it is logically equivalent to infinite descent.

For mathematical induction, build from the ground up. In order to show that every number n is either even or odd, we start with the base case when $n = 1$. We know 1 is odd, so the statement is true for $n = 1$. Next, we show how to argue that if a statement is true for one value of n , it will

be true for the next value of n . That's called the inductive step. But we've already done that when we showed that if a number was even or odd, then the next number was odd or even. The principle of mathematical induction says that if the statement is true for 1, and the inductive step is valid, then it's true for all numbers.

Axiom. If a property of numbers holds for $n = 1$, and if it holds for any number n it also holds for $n + 1$, then it holds for all numbers.

We can write this axiom symbolically if we add some new notation. Let $S(n)$ be an abbreviation for "the property S holds for n ." In what we're looking at $S(n)$ means " n is either even or odd." Let $\forall n$ be an abbreviation for "for all n ." That's called a universal quantifier. Now we can write the above axiom of mathematical induction as

If $S(1)$, and $\forall n (S(n) \text{ implies } S(n + 1))$, then $\forall n S(n)$.

Minimization. A third principle, equivalent to minimization and mathematical induction, is called minimization.

Axiom. If a property of numbers holds for a least one number, then it holds for a smallest number.

Minimization is usually used in proofs by contradiction. Here's how we can use it in a proof by contradiction to show every number is either even or odd.

Assume that not every number is either even or odd. There there is some number which is neither even nor odd. By the principle of minimization, it follows that there is a least number n which is neither even nor odd. Now, this n can't be 1, since 1 is odd, so n has a predecessor $n - 1$. Since n is the smallest number that is neither even nor odd, therefore $n - 1$, being a smaller number than n , must be either even or odd. But that implies its successor n is either odd or even, a contradiction. Since the assumption that not every number is either even or odd leads to a contradiction, we may conclude that every number is either even or odd.

Generally speaking, proofs by contradiction are a little harder to follow than direct proofs, but, generally speaking, it's easier to come up with a proof

by contradiction than it is to come up with a direct proof.

We did it! We've got three explanations why every number is either even or odd. In the process we found definitions for even and odd, a couple other theorems, two basic axioms about numbers—1 doesn't have a predecessor, any number except 1 does have a predecessor—and three choices for a third axiom—infinite descent, mathematical induction, and minimization. We could take any one of the three choices as the third axiom, then prove the other two follow from it. We won't do that now.

Loose ends. Still, there's a loose end to tie up. We've still got to prove that theorem that we said we'd prove later. Also, when we say every number is either even or odd, we haven't explicitly said "but not both." Let's see if we can show no number is both even and odd. We already know that the number 1 is odd. We'll need to know it's not even.

Theorem. Every number other than 1 does have a predecessor, that is, if $n \neq 1$, then $\exists m : n = m + 1$.

Proof. (Delayed from earlier.) We'll prove the logically equivalent statement

For each number n , either $n = 1$ or n has a predecessor

by induction.

The base case holds since if $n = 1$ then either $n = 1$ or n has a predecessor.

Now assume the statement is true for n . We'll show it's true for $n + 1$, that is, either $n + 1 = 1$ or $n + 1$ has a predecessor. But $n + 1$ has the predecessor n . Q.E.D.

Theorem. The number 1 is not even.

Proof. Suppose 1 is even. Then $1 = m + m$ for some number m . If m is 1, then $1 = m + 1$ which says that 1 is the successor of some number, which it isn't. But if m is not 1, then it is a successor itself, $m = k + 1$, but then $1 = m + k + 1$ says that 1 is again the successor of some number, which it isn't. Thus, the assumption that 1 is even leads to a contradiction. Therefore, 1 is not even. Q.E.D.

We'll also need the converse to theorem above that says that the number after an odd number is

even, and the number after an even number is odd.

Theorem. If a number has a predecessor, then when that number is even its predecessor is odd, but if that number is odd its predecessor is even.

Proof. Let the predecessor be n so that the number is $n + 1$. Suppose that the number $n + 1$ is even, that is $n + 1 = m + m$ for some m . Then by our definition for odd, that says the predecessor n is odd.

Now suppose that the number $n + 1$ is odd. Then $n + 1 + 1 = m + m$ for some m . First note that m is not 1, for if $m = 1$, then $n + 1 + 1 = 1 + 1$, which implies $n + 1 = 1$, that is, 1 has a predecessor, which it doesn't. Therefore m is not 1, and, thus, has its own predecessor, $m = k + 1$. Then $n + 1 + 1 = k + 1 + k + 1$, so $n = k + k$, which says n is even. Q.E.D

Now we can prove the theorem.

Theorem. No number is both even and odd.

Proof. We can arrange the proof so that we use induction, but minimization is just a little simpler here.

Suppose that some number is both even and odd. Then there is a smallest number n that is both even and odd. This number n is not 1, since we've already proved that 1 is not even. Therefore n must have a predecessor. But then, since n is both even and odd, by the previous theorem, then its predecessor will be both odd and even, and that contradicts the minimality of n . Thus, there can't be a smallest number that's both even and odd, so there can't be any number that's both even and odd. Q.E.D

We can summarize our investigation on even and odd numbers—their parity—in a few sentences. Every number has a parity, that is, it is either even or odd, but not both. The number 1 is odd. The successor of a number has the opposite parity, also, the predecessor of a number, if it has one, has the opposite parity.

History. Parity is one of the oldest concepts in formal mathematics. Euclid devotes part of Book IX of his *Elements* to the concept. He includes propositions such as these three.

Proposition IX.21. If as many even numbers as we please are added together, then the sum is even.

Proposition IX.26. If an odd number is subtracted from an odd number, then the remainder is even.

Proposition IX.29. If an odd number is multiplied by an odd number, then the product is odd.

Each of his propositions on even and odd numbers comes with a proof. His definition for even and odd numbers appear earlier in Book VII, the first book in the *Elements* on number theory.

Definition VII.6. An *even number* is that which is divisible into two equal parts.

Definition VII.7 An *odd number* is that which is not divisible into two equal parts, or that which differs by a unit from an even number.

His definition of even number is the same as ours, but he includes two clauses in his definition of odd number. From his use of this definition in various propositions, it's clear that he means them to be equivalent, that is, he assumes that a number is not even if and only if it differs from an even number by 1. But he never proves that equivalence and he never made that an explicit axiom.

The first few books of the *Elements* are about plane geometry and Euclid includes many axioms for geometry. But he has no axioms for number theory, and it's unclear why. Perhaps he didn't think axioms were necessary for numbers. More likely, he just didn't devote as much time to the foundations of number theory as he did for the foundations of geometry. It appears that he restructured plane geometry into a logical order, but he didn't do the same for number theory. The propositions stated above from Book IX on even and odd numbers are very elementary, but they don't come at the beginning of the discussion of number theory, which begins in Book VII, but are inserted in the last third of Book IX. He might have left them out altogether if they weren't needed to prove his last proposition in Book IX.

Proposition IX.36. If as many numbers as we please beginning from a unit are set out continuously in double proportion until the sum of all be-

comes prime, and if the sum multiplied into the last makes some number, then the product is perfect.

(A number is *perfect* if it is the sum of its proper divisors, Definition VII.22.) It is likely that the propositions about even and odd numbers were taken from some earlier work on number theory, now lost, as a group, and Euclid did not critically analyze them as he did for his propositions on geometry. Indeed, it is likely that these propositions on even and odd numbers are among the earliest propositions that appear in the *Elements*, and that go back a century or two before Euclid to the Pythagoreans.