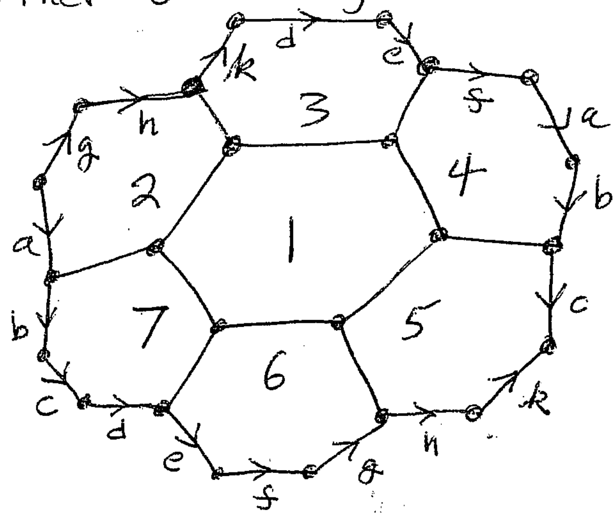
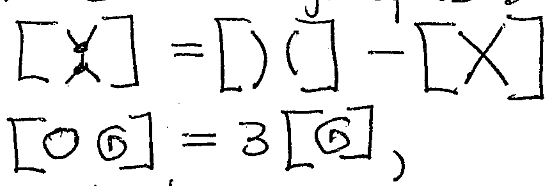
Graph Theory Problem Set

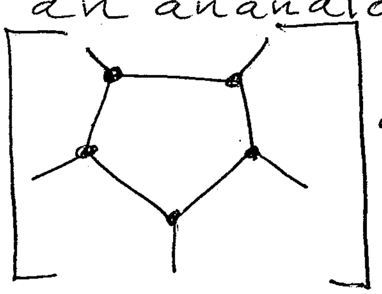
1. Show that the identification space indicated below gives a torus paved by 7 hexagons such that each hexagon shares edges with each of the other 6 hexagons.



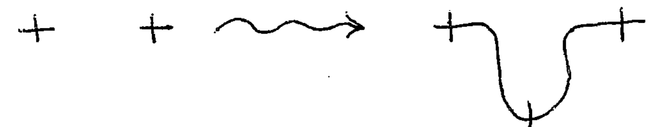
2. Using the Penrose coloring formula for plane cubic graphs:



prove that 

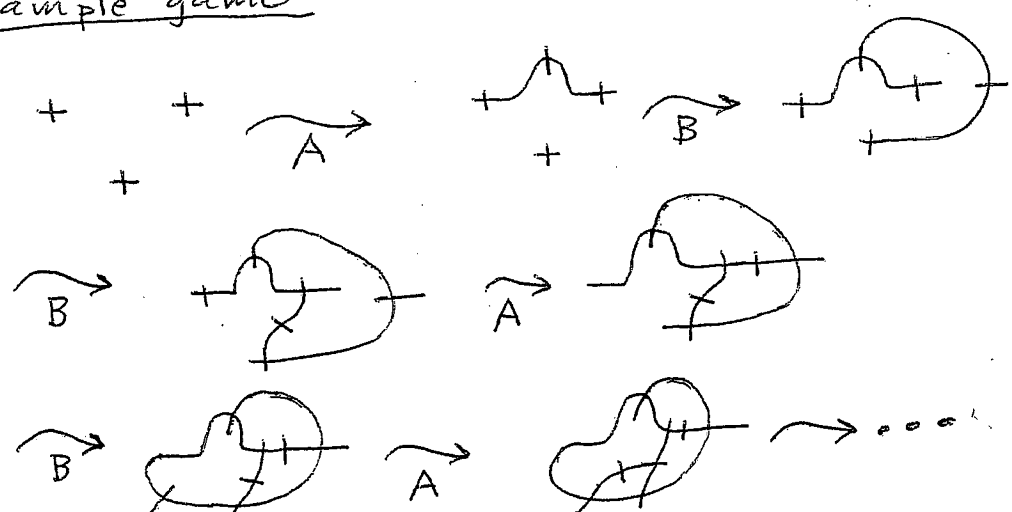
Find an ananalogous formula for .

3. Brussels Sprouts

- Start with a number of crosses: +
- A move: 

(The connection sprouts a new cross. The new edge does not intersect any previous edges.)

- sample game



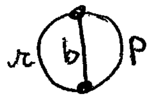
(A) Show that a game with n crosses will end in a finite number of steps.

(B) Show that a game with n crosses will have $\underline{F(n)}$ moves. Find the function $F(n)$.

(C) In the above, you analyzed Brussels sprouts for games in the plane. Generalize your results to

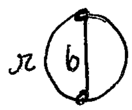
(a) orientable surfaces of genus g.

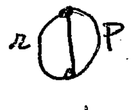
(b) the projective plane.

4. A cubic graph (3 edges per node) is said to be <u>edge 3-colored</u> if each edge receives one of three colors (r, b, p) and every vertex sees three distinct colors. For example, 

is edge 3-colored.

Consider <u>two-color circuits</u> in such a coloring. They can be of the form rb, rp or bp. For example



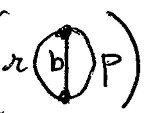
is a rb circuit, 

is an rp circuit and 

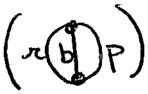
is a bp circuit.

Define the <u>parity</u> $\pi(G, \alpha)$ [where α denotes an edge 3-coloring of G] (0 if even, 1 if odd) to be the parity of the <u>sum</u>

$$S(G,\alpha) = \#(\text{rb circuits}) + \#(\text{rp circuits}) + \#(\text{bp circuits}).$$

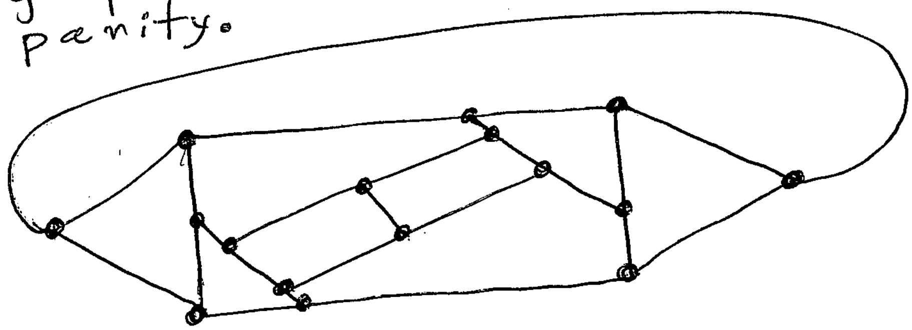
Thus π(

) = 1 (odd) since

S(

) = 3.

Find <u>two</u> edge 3-colorings for the graph <u>below</u> that have opposite parity.



5. (a) Construct an embedding (tight) of K_6 in the torus.

(b) Prove that for a planar drawing of K_6 with crossings one needs $\lambda = 9$ for the λ-curve method to produce an embedding of K_6 in the torus.

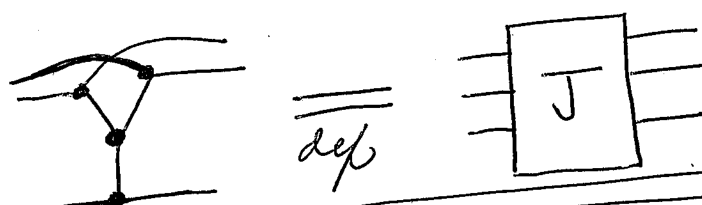
(c) Find a planar drawing of K_6 with crossings and $\lambda = 9$.

6. Recall the definition of parity $\pi(G, \alpha)$ for α an edge 3-coloring of cubic graph G. Let C be any single 2-color circuit for (G, α). Let α' be the coloring of G obtained by switching the colors (e.g., $r \to b$ & $b \to r$) on C.

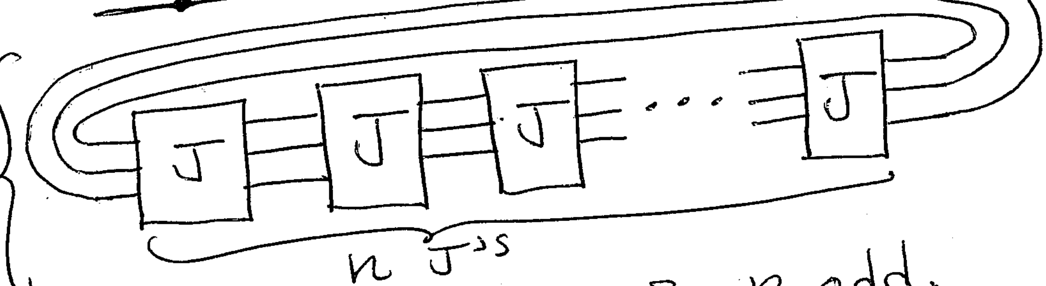
Claim $\pi(G, \alpha) = \pi(G, \alpha')$ whenever G is planar.

Investigate this claim with examples. Prove it if you can.

7.



$J^{(n)}$



Prove that for $n \geq 3$, n odd, $J^{(n)}$ is not edge 3-colorable.