Math 313 — HW#1 — Problems on pages 11 & 12.

1.2.1 | a) Suppose $\sqrt{3} = P/Q$, $P, Q \in N = \{1, 2, \ldots\}$ no common divisor

then $3 = P^2/Q^2 \Rightarrow 3Q^2 = P^2 \Rightarrow 3 \mid P^2$.

Note that numbers mod 3 are of the form $3K, 3K+1, 3K+2$. $(3K)^2 = 9K^2 = 3(3K^2)$,

$(3K+1)^2 = 9K^2 + 6K + 1 = 3(3K^2 + 2K) + 1$,

$(3K+2)^2 = 9K^2 + 12K + 4 = 3(3K^2 + 4K + 1) + 1$.

Thus $3 \mid P^2 \Rightarrow 3 \mid P$.

Hence $P = 3P'$ & $\therefore 3Q^2 = (3P')^2 = 9P'^2$

$\Rightarrow Q^2 = 3P'^2 \Rightarrow 3 \mid Q^2 \Rightarrow 3 \mid Q$. We have shown that $3 \mid P \wedge 3 \mid Q$. This is a contradiction. Therefore $\sqrt{3}$ is not rational.

a) The same argument does work to show $\sqrt{6}$ irrat. It will run: If $\sqrt{6} = P/Q$ reduced then $\sqrt{6}^2 = P^2/Q^2 \Rightarrow 6Q^2 = P^2 = 6 \mid P^2 \Rightarrow$

$2 \mid P^2 \wedge 3 \mid P^2 \Rightarrow 2 \mid P \wedge 3 \mid P \Rightarrow 6 \mid P$ etc...

b) If we try to suppose $\sqrt{4} = P/Q$ & square to $4 = P^2/Q^2 \Rightarrow 4Q^2 = P^2$

$\Rightarrow 4 \mid P^2$. But $4 \mid P^2 \not\Rightarrow 4 \mid P$.

e.g. $4 \mid 2^2$ but $4 \nmid 2$. So the argument does not work.

1.2.2 | (a) False. Let $A_n = [0, \frac{1}{n}] = \{x \mid 0 \leq x \leq \frac{1}{n}\}$

$n = 1, 2, 3, \ldots$. Then $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$

but $\bigcap_{n=1}^{\infty} A_n = \{0\}$ is a finite set.

(b) True. Proof: Suppose that $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

We will show this leads to a contradiction. (next page)

Let $x \in A_1$. (We are assuming $\bigcap_{n=1}^{\infty} A_n = \phi$)

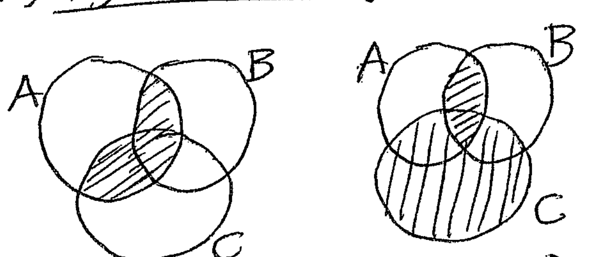
Then $x \notin A_K$ for some $K > 1$, because if $x \in A_K \ \forall K > 1$ then $x \in \bigcap_{n=1}^{\infty} A_n$ + this is ϕ.

Let $\lambda(x)$ = the <u>least</u> K s.t. $x \notin A_K$.

Now let m = the <u>largest</u> of the numbers $\lambda(x)$ for $x \in A_1$ (A_1 is finite and so there is a largest such number.). Then it follows that for every $x \in A_1$, $x \notin A_m$ and hence $x \notin A_n$ for all $n > m$. ($A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$).

This implies that $A_n = \phi \ \forall n \geq m$ + hence a <u>contradiction</u>, since we were given that each $A_n \neq \phi$ and finite. //

(c) <u>False</u>: (i) <u>By Venn Diagrams</u>



$$A \cap (B \cup C) \neq (A \cap B) \cup C$$

(ii) $A = \{0, 1, 2, 3\}$, $B = \{1, 2, 4, 5\}$

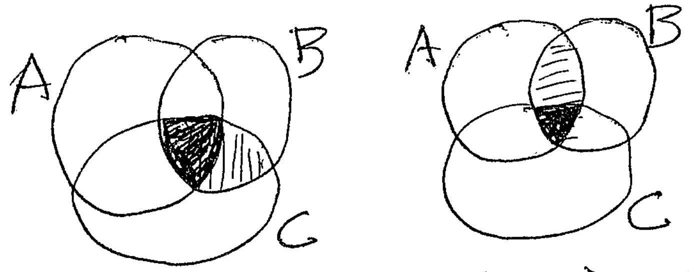
$C = \{3, 5, 6\}$

$A \cap (B \cup C) = \{0, 1, 2, 3\} \cap \{1, 2, 3, 4, 5, 6\}$

$= \{1, 2, 3\}$ $\neq$

$(A \cap B) \cup C = \{1, 2\} \cup \{3, 5, 6\} = \{1, 2, 3, 5, 6\}$.

(d)

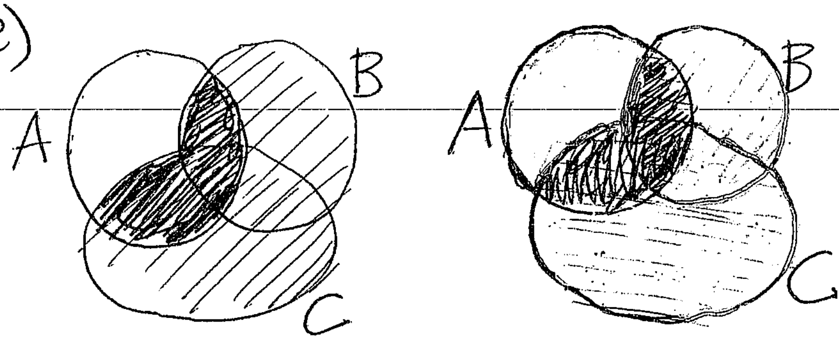


True.

(very dark shaded parts)

$$A\cap(B\cap C) = (A\cap B)\cap C$$

(e)



True

$A\cap(B\cup C)$ $\qquad$ $(A\cap B)\cup(A\cap C)$

Or: $x\in A\cap(B\cup C)$

$$\iff x\in A \wedge (x\in B \vee x\in C)$$

$$\iff (x\in A \wedge x\in B) \vee (x\in A \wedge x\in C)$$

$$\iff (x\in A\cap B) \vee (x\in A\cap C)$$

$$\iff x\in (A\cap B)\cup(A\cap C).$$

1.2.3 (a) (b) & (c)

$$x\in(A\cap B)^c \iff x\notin A\cap B$$

$$\iff x\notin A \vee x\notin B$$

$$\iff x\in A^c \vee x\in B^c$$

$$\iff x\in(A^c\cup B^c)$$

$$\therefore \boxed{(A\cap B)^c = A^c\cup B^c}$$

1.2.4

> Δ inequality says that for all real numbers a, b we have
> $$|a+b| \leq |a| + |b|.$$

a) Suppose $a \geq 0$ & $b \geq 0$.

Then $|a+b| = a+b = |a|+|b|$ ✓

Suppose $a<0$ & $b<0$. Then

$|a| = -a$, $|b| = -b$ and $|a+b| = -(a+b)$.

(since $a<0$ & $b<0 \Rightarrow a+b<0$).

$$\therefore |a+b| = -(a+b) = (-a) + (-b) = |a| + |b|$$ ✓

(b) Suppose $a \geq 0$, $b<0$ & $a+b \geq 0$.

Then $|a+b| = a+b = |a|+b$

$|b| = -b$ & since $b<0$,

$|b| > b$.

$\therefore |a+b| = |a|+b < |a|+|b|$ ✓ °

1.2.5 (a) $|a-b| = |a+(-b)| \leq |a| + |-b| = |a|+|b|$

$\therefore |a-b| \leq |a|+|b|$ ✓

(b) Show that $||a|-|b|| \leq |a-b|$.

<u>Proof.</u> If a & b have the same sign, then this is an equality. So suppose $a \geq 0$ and $b<0$. (Check that proving for this case is all we have to do.)

Then $||a|-|b|| = |a+b|$

$|a-b| = |a+|b||$

Let $r = |b|$.

(next page)

Thus we must show that

$$|a-b| \geq \big||a|-|b|\big|$$

$$|a-b| = |a+r|, \qquad \big||a|-|b|\big| = |a-r|$$

when $a \geq 0$, $r \geq 0$.

We must show that $|a-r| \leq |a+r|$.

But $|a+r| = |a|+|r|$.

& so we must show

$$|a-r| \leq |a|+|r|$$

This *is* the Δ-inequality. //

1.2.6 $f: X \to Y$, $A \subseteq X$, $f(A) = \{f(x) \mid x \in A\}$.

(a) $f(x) = x^2$, $A = [0,2]$, $B = [1,4]$

$f(A) = [0,4]$, $f(B) = [1,16]$

$f(A \cap B) = f([1,2]) = [1,4]$

$f(A) \cap f(B) = [0,4] \cap [1,16] = [1,4]$

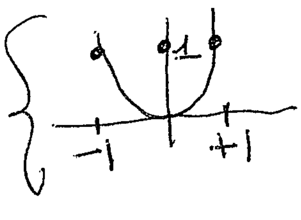
$f(A \cup B) = [0,16]$

$f(A) \cup f(B) = [0,16]$

(b) Give an example of A, B s.t. $f(A \cap B) \neq f(A) \cap f(B)$.

Note: $x \in f(A \cap B) \Rightarrow x \in f(A) \wedge x \in f(B) \Leftarrow x \in f(A) \cap f(B)$ (one way arrow!) ($a \neq b$ possible)

$x \in f(A) \cap f(B) \Rightarrow x = f(a) \wedge x = f(b)$



$A = \{-1\}$, $B = \{+1\}$: $f(A \cap B) = \emptyset$, $f(A) \cap f(B) = \{1\}$

(c) $g : \mathbb{R} \longrightarrow \mathbb{R}$, $A, B \subseteq \mathbb{R}$

$$x \in g(A \cap B) \iff x = g(c),\ c \in A \cap B$$
$$\implies x = g(c), c \in A \wedge x = g(c), c \in B$$
$$\implies x \in g(A) \wedge x \in g(B)$$
$$\implies x \in g(A) \cap g(B)$$

$\therefore$ $g(A \cap B) \subseteq g(A) \cap g(B)$

An element x of $g(A) \cap g(B)$ but not in $g(A \cap B)$ can happen if $g(a) = x = g(b)$ but $a \notin A \cap B$ & $b \notin A \cap B$.

(d) $x \in g(A \cup B) \iff x \in g(A) \vee x \in g(B)$
$$\iff x \in g(A) \cup g(B)$$

$\therefore$ $g(A \cup B) = g(A) \cup g(B)$

1.2.8 (a) $\sim [\forall a, b \in \mathbb{R} .\ a < b\ \exists n \in N .\ a + \frac{1}{n} < b]$

$\|$

$\exists a, b \in \mathbb{R} .\ a < b .\ \sim \exists n \in N .\ a + \frac{1}{n} < b$

$\|$

$\exists a, b \in \mathbb{R} .\ a < b .\ \forall n \in N .\ a + \frac{1}{n} \geq b .$

(b) $\sim\left[\forall a,b \in \mathbb{R} .\ a<b,\ \exists r \in \mathbb{Q} .\ a<r<b\right]$

$$\|$$

$\exists a,b \in \mathbb{R} .\ a<b,\ \forall r \in \mathbb{Q},\ \sim(a<r<b).$

(c) $\sim\left[\forall n \in N,\ \sqrt{n} \in N \cup \mathcal{I}\right]$ ($\mathcal{I}$ = irrats)

$$\|$$

$\exists n \in N,\ \sqrt{n} \notin (N \cup \mathcal{I})$

$$\|$$

$\exists n \in N,\ \sqrt{n} \in (N \cup \mathcal{I})^c$

$$\|$$

$\exists n \in N,\ \underbrace{\sqrt{n} \in N^c \cap \mathcal{I}^c}$

"$\sqrt{n}$ is not a natural number and $\sqrt{n}$ is not irrational."

(d) $\sim\left[\forall x \in \mathbb{R},\ \exists n \in N .\ n > x\right]$

$$\|$$

$\exists x \in \mathbb{R},\ \forall n \in N .\ n \leq x$

1.2.9 $x_1 = 1$, $x_{n+1} = \left(\frac{1}{2}\right)x_n + 1$

Show: $x_n \leq 2 \ \forall n \in N$

Pf. I. $n=1$. $x_1 = 1 < 2$ ✓

II. Suppose $x_k < 2$. $\left(\Rightarrow \frac{x_k}{2} < 1\right)$

Then $x_{k+1} = \frac{x_k}{2} + 1 < 1 + 1 = 2$. //

Note: $x_2 = \frac{1}{2} + 1 = 3/2$ $x_4 = \frac{7}{8} + 1 = 15/8$

$x_3 = \frac{3}{4} + 1 = 7/4$ $x_5 = \frac{15}{16} + 1 = \frac{31}{16}$

...

You can prove that $\boxed{x_n = \frac{2^n - 1}{2^{n-1}}}$.

And then you can show ~~that~~

$\boxed{\lim_{n \to \infty} x_n = 2}$.

Note: $\boxed{2 - x_n = 1/2^{n-1}}$

1.2.10 $y_1 = 1$, $y_{n+1} = (3y_n + 4)/4$.

(a) I. $y_1 = 1 < 4$ ✓

II. Suppose $y_k < 4$ some k.

Then $y_{k+1} = \frac{3}{4}y_k + 1 < \frac{3 \cdot 4}{4} + 1 = 4$ ✓ //

(b) I. $y_2 = \frac{3+4}{4} = 7/4 > 1 = y_1$.

II. Suppose $y_k > y_{k-1}$.

$y_{k+1} = \frac{3}{4}y_k + 1 \Rightarrow y_{k+1} - y_k = -\frac{y_k}{4} + 1 > 0$ ✓ //

$y_k < 4$
$\Rightarrow 0 < 4 - y_k$
$\Rightarrow 0 < 1 - y_k/4$

1.2.11 $P(A)$ = set of subsets of A.

Show for $|A| = n$ that $|P(A)| = 2^n$.

Proof. Consider functions
$$f : A \longrightarrow \{0,1\}.$$
~~Let $\mathcal{F}(A)$ = all such functions.~~

Since every $f \in \mathcal{F}(A)$ is a choice of 0 or 1 for each elt of A, it is clear (prove it!) that $|\mathcal{F}(A)| = 2^n$.

Define $F : \mathcal{F}(A) \longrightarrow P(A)$
$$F(f) = \{a \in A \mid f(a) = 1\}.$$
Define $G : P(A) \longrightarrow \mathcal{F}(A)$ by
$$G(S)(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$
Check that $\mathcal{F}(A) \underset{G}{\overset{F}{\rightleftarrows}} P(A)$ is a 1-1 correspondence.

$\therefore |P(A)| = |\mathcal{F}(A)| = 2^n$. //

1.2.12 We know from before that
$(A \cup B)^c = A^c \cap B^c$ (DeMorgan)
for any sets A, B.

(a) I. $n=1$. $A_1^c = A_1^c$ ✓
$n=2$. $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$ ✓

II. Suppose
$(A_1 \cup \cdots \cup A_k)^c = A_1^c \cap \cdots \cap A_k^c$
for some k.

Then $(A_1 \cup \cdots \cup A_k \cup A_{k+1})^c$
$= ([A_1 \cup \cdots \cup A_k] \cup A_{k+1})^c$
$= [A_1 \cup \cdots \cup A_k]^c \cap A_{k+1}^c$ (DeMorgan)
$= (A_1^c \cap \cdots \cap A_k^c) \cap A_{k+1}^c$ (Induction Hypothesis)
$= A_1^c \cap A_2^c \cap \cdots \cap A_{k+1}^c$ //

(b) There is only <u>one</u> single proposition to prove.

(c) <u>Yes. Valid.</u> $x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c \Leftrightarrow x \notin \bigcup_{n=1}^{\infty} A_n$
$\Leftrightarrow \forall n \in \mathbb{N}, x \notin A_n \Leftrightarrow \forall n \in \mathbb{N}, x \in A_n^c$
$\Leftrightarrow x \in \bigcap_{n=1}^{\infty} A_n^c$ //

Extra Problem.

Show $\lim_{n\to\infty} S_n = \frac{1}{1-a}$ if $|a|<1$

when $S_n = 1+a+a^2+\cdots+a^n$.

Proof. Check by induction that

$$S_n = 1+a+a^2+\cdots+a^n = \frac{1-a^{n+1}}{1-a}.$$

Thus $\left|\frac{1}{1-a} - S_n\right| = \left|\frac{a^{n+1}}{1-a}\right|$. We will assume $0<a<1$ and leave proof for $a<0$ to you. Then $\left|\frac{a^{n+1}}{1-a}\right| = \frac{a^{n+1}}{1-a}$ and we want $\frac{a^{n+1}}{1-a} < \varepsilon$ for some given $\varepsilon > 0$. Thus

$$a^{n+1} < \varepsilon(1-a) \Longrightarrow (n+1)\log(a) < \log(\varepsilon(1-a)).$$ Then

& note $\log(a) < 0$ since $0<a<1$. & so need

$(n+1)|\log(a)| > |\log(\varepsilon(1-a))|$ &

$(n+1) > \frac{|\log(\varepsilon(1-a))|}{|\log(a)|}$ whence $n > \left(\frac{|\log(\varepsilon(1-a))|}{|\log(a)|} - 1\right)$.

Whenever n is this large, we have $\left|\left(\frac{1}{1-a}\right) - S_n\right| < \varepsilon$. This shows that $\lim_{n\to\infty} S_n = \frac{1}{1-a}$. //

Discussion. Imagine infinite binary numbers. So $111 = 2^2+2^1+2^0$ and $1111 = 2^3+2^2+2^1+2^0$. An infinite number would have the form:

$\ldots 11111 = \ldots + 2^4+2^3+2^2+2^1+2^0 = \Omega$.

Then $\Omega + 1$:

$$\begin{array}{r} \ldots 11111 \\ 1 \\ \hline \ldots 00000 \end{array}$$

← carry indefinitely!

$\Omega + 1 = 0$. i.e.

So it seems we have $\ldots + 2^4+2^3+2^2+2^1+2^0 = -1$ in this interpretation. Note that if we ignore $|a|<1$ we would have $\frac{1}{1-2} = -1$ & "$1+2+2^2+2^3+\cdots = \frac{1}{1-2}$ $= -1$". Comments??