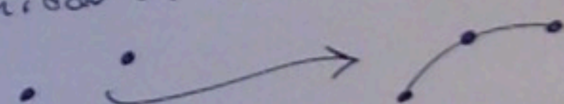
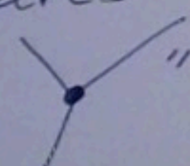
Math 467 — Homework

1. Recall the rules for sprouts:
   (a) start with n dots in the plane. e.g. $n=3$

   (b) A legal move connects two dots and puts a dot in the middle of the connecting arc.



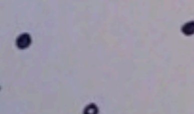
   (c) No dot can have more than 3 arcs touching it.



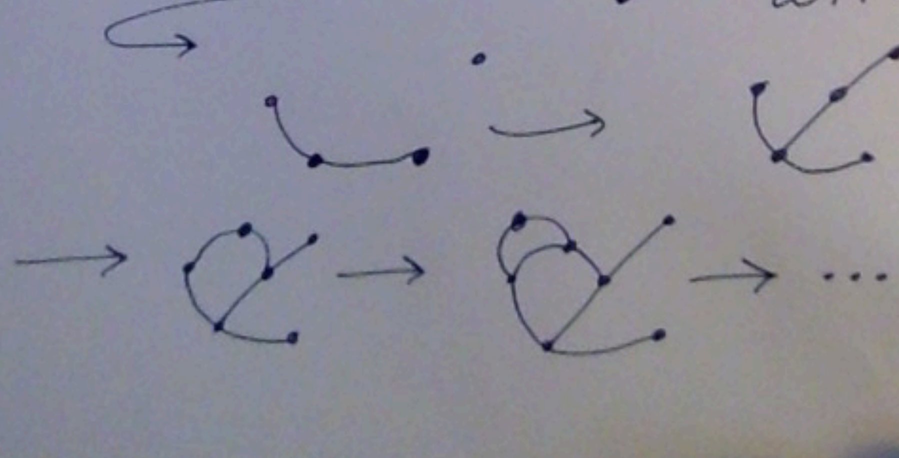
"max".

   (d) Winner makes the last move.

For example:

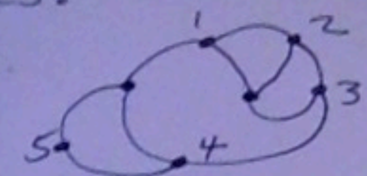


start with 3



(i) Play a number of 2 dot games.

e.g



1, 2, 3, 4, 5 are the orders of moves (first, second, third, ...). Here 1^{st} player won.

Who should win the 2 dot game?

Notice that in the game above there were 5 moves.

$5 = 2 \times 3 - 1$.

(ii) Give an example of a 3 dot sprouts game that has $3 \times 3 - 1 = 8$ moves.

(iii) I claim that the longest possible n dot sprouts game has $(3n-1)$ moves.

Can you give an example of such a game for any given value of n?

Explore examples.

2. A perfect number n is a number that is equal to the sum of all its divisors ($\neq$ itself). For example, $6 = 1+2+3$ is perfect because the numbers that divide $6 = 1\times 2 \times 3$ are 1, 2 and 3. ③

(i) Show that 28 is a perfect number.

(ii) Show that 496 is a perfect number.

(iii) Euclid proved that if $(2^p - 1)$ is a prime number, then $N = 2^{(p-1)}(2^p - 1)$ is perfect. For example,

$$6 = 2 \times 3 = 2^{(2-1)}(2^2 - 1)$$

and

$$28 = 2^2 \times 7 = 2^2 \times (2^3 - 1)$$

Show that 496 also has this form for $p = 5$.

(iv) Find a perfect number larger than 496.

3. A prisoner has been given the following task. Place one grain of sand on the first square of a chess board, 2 grains on the second square, 4 grains on the third square, ... and so on until you put 2^{64} grains on the last square.

Thus the prisoner must place $N = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{64}$ grains of sand on the chessboard.

(i) I claim that

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{64} = 2^{65} - 1.$$

Can you explain why this is so?

(ii) Assuming that the prisoner places one grain per second on the chessboard, how many years will it take for him to place $(2^{65} - 1)$ sand grains on the board?

(iii) Max found the following "proof" in a very old book.

Theorem. $-1 = (1+2+2^2+2^3+2^4+\cdots)$

Proof. Let $S = 1+2+2^2+2^3+\cdots$.

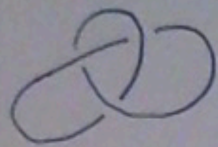
$$S = 1+2+4+8+16+\cdots$$
$$\Rightarrow S = 1+2(1+2+4+8+\cdots)$$
$$\Rightarrow S = 1+2S$$
$$\Rightarrow 2S+1 = S$$
$$\Rightarrow S+1 = 0$$
$$\Rightarrow S = -1.$$

Thus $-1 = 1+2+4+8+16+\cdots$ QED

Max said "Well I would say that $\infty = 1+2+4+8+16+\cdots$ So this manuscript shows that **Infinity Equals Minus One**. I do not know what this means, but it must be very deep."

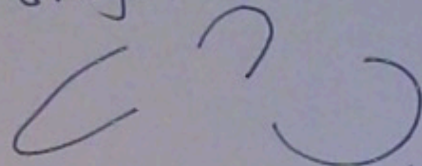
Write a dialogue between yourself and Max discussing this "proof".

4. This is a "knot diagram":

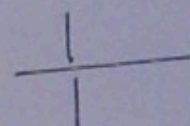


<u>The trefoil knot.</u>

Knot diagrams are composed of arcs

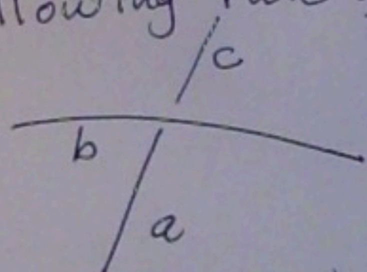


put together so that there are <u>crossings</u>



where the arcs meet.

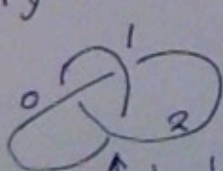
I "color" a knot diagram with numbers by the following rule:



$\Rightarrow c = 2b - a.$

Each arc receives a color.

Try to color the trefoil:



$\boxed{2\times 1 - 0 = 2}$

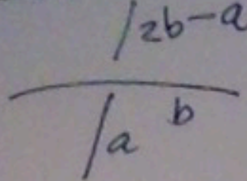
↑ but $2\times 2 - 1 = 3$

So I need $3 = 0$.

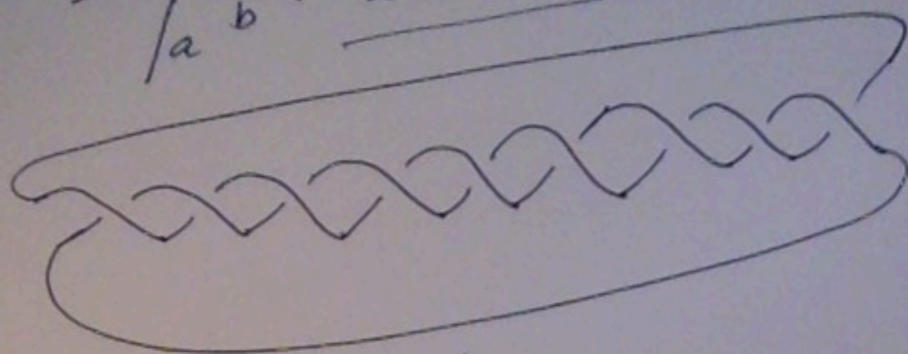
OK! $3 \equiv 0 \bmod 3$.

We color <u>mod3</u> using colors $\{0,1,2\}$.

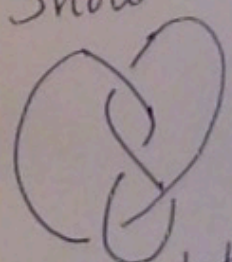
(i) Color this knot using



and in <u>mod 3</u>.



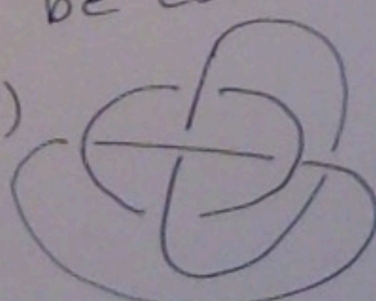
(ii) Show that



can be colored <u>modulo 5</u>.

and that it cannot be colored <u>modulo 3</u> (with more than one color)

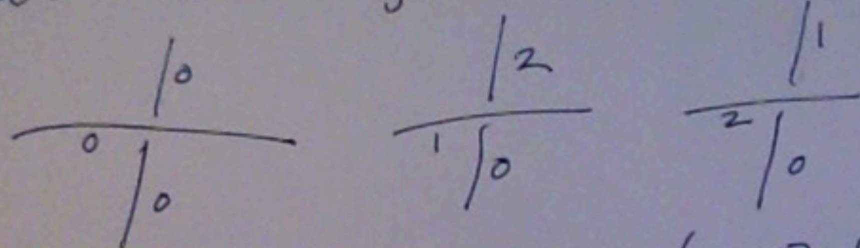
(iii)



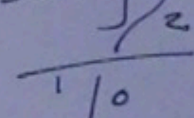
Show that these rings can not be colored with more than one color <u>modulo 3</u>.

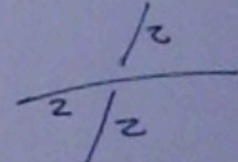
(8)

(iv) Note that for mod 3 colorings we have the possibilities at crossings:



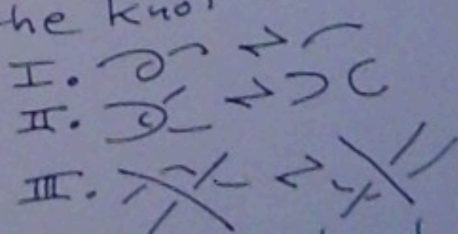
$(2 \times 2 - 0 = 4 \equiv 1 \bmod 3)$

Check that mod 3 coloring is the same as satisfy the rule that every crossing has either 3 colors (e.g. )

or 1 color. (e.g. )

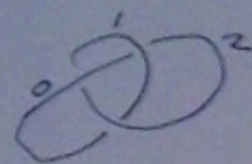
$(2 \times 2 - 2 = 4 - 2 = 2)$

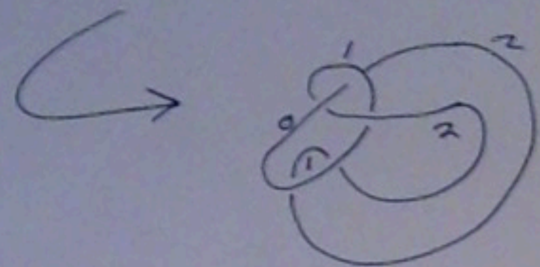
(v) You can see that if you move the knot around by moves like

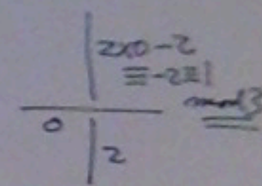


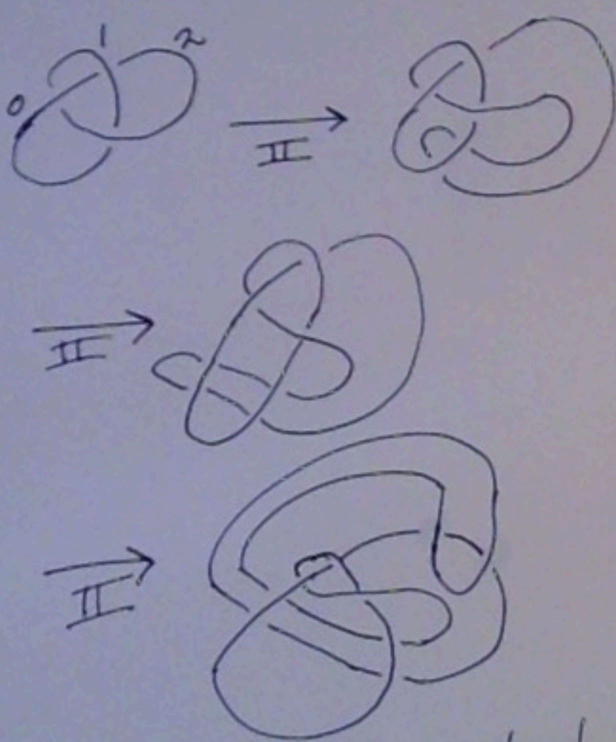
then if one diagram is colored then so will the other one.

e.g.









Use this method to induce colorations modulo 3 on each of the above knots.

5. Recall <u>Wilson's Theorem:</u> (10)

p is prime if and only if p divides $(p-1)!+1$.

<u>Verify by direct calculation that 13 divides $12!+1$.</u>

<u>Note</u>: This means you want to show that $12! \equiv -1 \bmod 13$.
Thus you want to calculate
$12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
<u>MODULO 13</u>.
<u>Start forming the products and reduce modulo 13 at each stage.</u>

[Here is an example for $6!+1$ mod 7:

$6 \times 5 \equiv 30 \equiv 2 \bmod 7$
$6 \times 5 \times 4 \equiv 2 \times 4 \equiv 8 \equiv 1 \bmod 7$
$6 \times 5 \times 4 \times 3 \equiv 1 \times 3 \equiv 3 \bmod 7$
$\Rightarrow 6! \equiv 3 \times 2 = 6 \equiv -1 \bmod 7$]