Solutions to Sample Exam 2 – Math 310

1. $A=\begin{bmatrix}1&2&1&3\\2&3&1&5\\3&4&1&7\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&1&3\\0&-1&-1&-1\\0&-2&-2&-2\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&-1&1\\0&1&1&1\\0&0&0&0\end{bmatrix}$

$$\begin{array}{cccc}\uparrow&\uparrow&\alpha&\beta\\x_1&x_2&&\end{array}$$

NS(A) = Null Space(A) consists in the vectors

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}\alpha-\beta\\-\alpha-\beta\\\alpha\\\beta\end{pmatrix}=\alpha\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}+\beta\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}$$

So NS(A) has basis $\mathcal{B}=\left\{\begin{pmatrix}1\\-1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}\right\}$.

$\dim NS(A)=2.$

(c) RS(A) = Row Space (A) has basis $\tilde{\mathcal{B}}=\{(1,0,-1,1),(0,1,1,1)\}$ and $\dim RS(A)=2.$

(a) Since 1st two cols of the row reduced form are pivot cols, $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}2\\3\\4\end{pmatrix}\right\}$ is a basis for Col(A). $\dim \operatorname{Col}(A)=2.$

(b) $\operatorname{Col}(A)^\perp=NS(A^T)$.

$A^T=\begin{bmatrix}1&2&3\\2&3&4\\1&1&1\\3&5&7\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&3\\0&-1&-2\\0&-1&-2\\0&-1&-2\end{bmatrix}\longrightarrow\begin{array}{c}\begin{array}{ccc}x_1&x_2&\alpha\end{array}\\\begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\\0&0&0\end{bmatrix}\end{array}:\begin{pmatrix}\alpha\\-2\alpha\\\alpha\end{pmatrix}=\alpha\begin{pmatrix}1\\-2\\1\end{pmatrix}$

$\Rightarrow \mathcal{B}''=\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}$ basis for $\operatorname{Col}(A)^\perp$ & $\dim \operatorname{Col}(A)^\perp=1$

2. (a) $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \\ y-x \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$\Rightarrow \quad M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} = [L]_{\mathcal{S}}^{\mathcal{S}'}$$

$\mathcal{S}$ = standard basis for $\mathbb{R}^2$

$\mathcal{S}'$ = standard basis for $\mathbb{R}^3$.

(b) $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ basis for $\mathbb{R}^2$.

$\mathcal{F} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ basis for $\mathbb{R}^3$.

$$E = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} : \quad F = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} :$$

$$\begin{tikzcd} \mathcal{E} \arrow[r, "E"] \arrow[d, "A = {[L]}_{\mathcal{E}}^{\mathcal{F}}"'] & \mathcal{S} \arrow[d, "M = {[L]}_{\mathcal{S}}^{\mathcal{S}'}"] \\ \mathcal{F} \arrow[r, "F"] & \mathcal{S}' \end{tikzcd}$$

$$\Rightarrow \quad \boxed{A = F^{-1} M E}$$

Calculate F^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}}_{F^{-1}}$$

$$A = F^{-1}ME$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 2 & -2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 4 \\ 2 & -2 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 0 & -2 \\ 0 & 4 \\ 2 & -2 \end{bmatrix}}$$

<u>Check</u>: This claims ~~that~~

$$L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

& $$L\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \checkmark$$

3. $V = \text{Span}\{e^x, xe^x\}$

$D : V \longrightarrow V$, $D = \frac{d}{dx}$.

(a) $W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}$

$= e^x(e^x + xe^x) - xe^{2x}$

$= e^{2x} + xe^{2x} - xe^{2x}$

$= e^{2x} \neq 0$

$\Longrightarrow \{e^x, xe^x\}$ lin ind.

(b) $De^x = e^x$

$D(xe^x) = e^x + xe^x$

$\beta = \{e^x, xe^x\}$

$[D]_\beta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

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4. $L : \mathbb{R}^2 \to \mathbb{R}^2$, $[L]_S = A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$

$E = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$, $\det E = -2 + 1 = -1 \neq 0$

$\Longrightarrow \beta = \left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$ basis for $\mathbb{R}^2$.

(5)

$$B = [L]_\beta^\beta \qquad E = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

$$E^{-1} = \frac{1}{(-1)} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\left.\begin{array}{ccc} \beta & \xrightarrow{E} & \alpha \\ \downarrow B & & \downarrow A \\ \beta & \xrightarrow{E} & \alpha \end{array}\right\} \quad B = E^{-1} A E$$

$$B = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ -3 & -6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\boxed{B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}$$

$$L\begin{pmatrix} 2 \\ -1 \end{pmatrix} = A\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 2\begin{pmatrix} 2 \\ -1 \end{pmatrix} \checkmark$$

$$A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -1 \end{pmatrix} \checkmark$$