DEGREES CODING MODELS OF ARITHMETIC

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INTRODUCTION

In 1959 Tennenbaum proved that if M is a nonstandard model of Peano arithmetic, (PA) then the addition and multiplication of M can not be recursive. This result leads to several natural questions:

i) Given T a complete extension of Peano arithmetic what Turing degrees code models of T?

ii) Given a model M of T, what Turing degrees code M?

This thesis begins with a survey of known results in this area including proofs of many folklore results. It also represents my contributions to the study of these problems.

In Section 1 we introduce Scott sets, the basic combinatorial tool used in studying degrees coding models of arithmetic, and begin to study the relationships between a model, the degrees that code it, its reals and its complete theory. Of the results in this section only 1.21 and 1.27 are new.

Section 2 is devoted to classification of degrees coding complete extensions of PA.

In Section 3 we consider what degrees code nonstandard models of $Th(\mathbb{N})$. We survey the results which led up to Solovay's classification theorem.

In Section 4 we give the proof of Theorem 3.5, which states that if every arithmetic set is recursive in d then there is a nonstandard model of $Th(\mathbb{N})$ recursive in d'. This is the main theorem of [M].

In Section 5 we give a classification of degrees coding S-saturated structures. We also review the necessary model theoretic preliminaries.

In Section 6 we apply the results of Section 5 to the study of degrees coding certain reducts and definable substructures of models of arithmetic.

In Section 7 we classify the degrees coding reducts and definable substructures of a fixed completion of PA. We also give an improvement of Solovay's classification of degrees coding models of $Th(\mathbb{N})$ and an application. Many of the results of Sections 5, 6 and 7 are joint work with Angus Macintyre and will appear in [Mac-M].

In Section 8 we study the degrees coding nonstandard models of a theory introduced to study the polynomial time hierarchy. We show that any such degree must be above $\underset{\sim}{0}'$.

Our model theoretic notation is standard. From recursion theory: $\{e\}^X$ denotes the e^{th} partial recursive function with oracle X, W_e^X denotes the domain of $\{e\}^X$, and if $d \subseteq \omega$, $\underset{\sim}{d}$ denotes the Turing degree of d. (Every Turing degree will be thought of as $\underset{\sim}{d}$ where $d \subseteq \omega$ is some cannonically chosen representative).

§1. SCOTT SETS

In this section we will lay the groundwork for our study of degrees coding models of arithmetic by examining the relationship between models of arithmetic and their associated Scott sets.

Throughout 2^{ω} will denote finite sequences of zeros and ones. There is a natural recursive coding of elements of 2^{ω} as elements ω. When no difficulties arise we will shamelessly confuse 2^{ω}, ω, and ω^n when it is convenient to do so. Similarly we confuse $P(\omega)$ and 2^{ω}.

<u>Definition 1.1</u>: $T \subseteq 2^{\omega}$ is a <u>tree</u> iff for any $\sigma \in T$ if $\tau \subseteq \sigma$, then $\tau \in T$. (Conforming to the above convention we will speak of $X \subseteq \omega$ as a tree if the subset of 2^{ω} which codes it is a tree.)

<u>Definition 1.2</u>: $S \subseteq P(\omega)$ is a <u>Scott set</u> iff

i) $X_1, \ldots, X_n \in S$ and $Y \leq_T X_1 \times X_2 \times \cdots \times X_n$, then $Y \in S$.

ii) If $T \in S$ is an infinite tree, then there is $f \in S$ an infinite path through T.

The next proposition gives us the most usefull property of Scott sets for model theorists. Let L be some fixed recursive language. We will identify L-sentence (or L-formulas) with elements of ω via Gödel codes.

<u>Proposition 1.3</u>: Let S be a Scott set. If $T \in S$ is an L-theory, then T has a completion $T^* \in S$.

<u>Proof</u>: Let $\varphi_1, \varphi_2, \ldots$ be a listing of all L-sentences. Let $X = \{\sigma \in 2^{\omega}$: for any $\tau \subseteq \sigma$, there is no proof of $\neg(\bigwedge_{\tau(i)=1} \varphi_i \wedge \bigwedge_{\tau(i)=0} \neg\varphi_i)$ from T with Gödel code at most $|\sigma|$ ($|\sigma|$ denotes the length of σ). X is recursive in T so $X \in S$. As S is a Scott

set there is an infinite path $f \in 2^{\omega} \cap S$. Clearly $T^* = \{\varphi_i : f(i) = 1\}$ is a completion of T. //

Scott sets arise naturally when studying nonstandard models of Pearo arithmetic (PA).

Definition 1.4: Let $M \models PA$. If $a \in M$ we define the real coded by a $r(a) = \{n \in \omega : M \models p_n | a\}$, where p_n denotes the n^{th} prime number. We let $\mathbb{R}e(M)$ denote $\{r(a) : a \in M\}$.

Proposition 1.5: (Scott-Tennenbaum [Sc-T]). If $M \models PA$ is nonstandard, then $\mathbb{R}e(M)$ is a Scott set.

Proof: Let $d \in M$ be some nonstandard element.

Claim 1: Suppose $a, b \in M$. Let $\sigma : M \to M^2$ be the usual recursive pairing function $a \mapsto \langle\sigma_0(a), \sigma_1(a)\rangle$. As M is a model of arithmetic $M \models \forall u\, v\ \exists w\ \forall i < d\ \ (p_i | w \iff (p_{\sigma_0(i)} | u \wedge p_{\sigma_1(i)} | v))$. Thus there is $c \in M$ such that for any $n \in \omega$ $M \models p_n | c \iff (p_{\sigma_0(n)} | a \wedge p_{\sigma_1(n)} | b)$. Thus $r(c)$ codes $r(a) \times r(b)$.

Claim 2: Suppose $a \in M$ and $A \leq_T r(a)$. Say $\{e\}^{r(a)}$ is the characteristic function of A. Let $B = \{b \in M : M \models p_b | a\}$. M believes B is finite. In M we can run computations $\{e\}^B$ for d steps. If we input a standard number n, then $\{e\}^B(n)$ will only make oracle queries to B about standard numbers and B will give affirmative answers iff $p_n | a$. So the computation $\{e\}^{r(a)}(n)$ will be identical to the computation $\{e\}^{\cancel{r(a)}\,B}(n)$. In particular, as $\{e\}^B(n)$ will halt at some standard stage, $n \in A$ iff $\{e\}^B(n)$ converges in d steps and outputs one. As $C = \{m \in M : m \leq d$ and $\{e\}^B(m)$ halts in d steps and outputs $1\}$ is M-finite, there is $b \in M$ such that for $n \leq d$ $p_n | b$ iff $n \in C$. Clearly $A = r(b)$.

<u>Claim 3</u>: Suppose $r(a)$ is an infinite tree T. Consider $T' = \{\sigma \in 2^{\underset{\smile}{M}}: p_\sigma | a\}$. For every standard n, there is $\sigma \in T'$ such that $|\sigma| = n$ and if $\tau \subseteq \sigma$, then $\tau \in T'$. By overspill there must be a nonstandard c and a $\sigma \in 2^{\underset{\smile}{M}}$ such that $|\sigma| = c$, $\sigma \in T'$ and every subsequence of σ is in T'. Every standard initial subseqaence of σ is in $2^{\underset{\smile}{\omega}}$ so it must be in T. Thus the standard initial subsequences of σ form an infinite path through T. In M we can find a b such that for any sequence $\tau \in 2^c$, $p_\tau | b$ iff $\tau \subseteq \sigma$. Clearly $r(b)$ is a path through T. //

Proposition 1.5 is the key model theoretic step toward Tennenbaum's result that no nonstandard model of arithmetic can be recursively presented. The next lemma is the key recursion theoretic step. Recall that two sets A and B are d-recursively inseparable iff $A \cap B = \emptyset$ and there is no d-recursive X such that $A \subseteq X$ and $B \cap X = \emptyset$. Given any set d the sets $A_d = \{n:\{n\}^d(n)$ converges with output $1\}$ and $B_d = \{n:\{n\}^d(n)$ converges with output $0\}$ are d-recursively inseparable d-r.e. . See Rogers [R] for details.

<u>Lemma 1.6</u>: If S is a Scott set and $d \in S$, then there is $X \in S$ such that $X \not\leq_T d$.

<u>Proof</u>: Let T be the tree $\{\sigma \in 2^{\underset{\smile}{\omega}}$: If $n \in \operatorname{dom} \sigma$ and $\{n\}^d(n)$ converges within $|\sigma|$ steps, then $\{n\}^d(n) = \sigma(n)\}$. Let $f \in S$ be an infinite path through T. Let $X = \{n: f(n) = 1\}$. Clearly $X \leq_T f$, so $X \in S$. As X separates A_d and B_d, $X \not\leq_T d$. //

In particular, every Scott set contains a nonrecursive element.

<u>Definition 1.7</u>: If M is a countable model of arithmetic we can think of M as $(\omega,\oplus,\odot)$. We define $\operatorname{Diag}(M)$ to be the set of atomic sentences

true in M. We will let deg(M) denote the Turing degree of Diag(M). Henceforth all models of arithmetic will be assumed countable and of the form $(\omega, \oplus, \odot)$. It is usefull to note that deg(M) is not an isomorphism invariant of M.

Lemma 1.8: If $M \models PA$ is nonstandard and $a \in M$, then $r(a) <_T \mathrm{Diag}(M)$.

Proof: First note that $n \in r(a)$ iff $\exists y\ \underbrace{y + \cdots + y}_{p_n\text{-times}} = a$ so $r(a)$ is r.e. in Diag(M). Further $n \notin r(a)$ iff

$$\bigvee_{0<i<p_n} \exists y\ \underbrace{y + y + \cdots + y}_{p_n\text{-times}} = a.$$

Thus $\{n : n \notin r(a)\}$ is r.e. in Diag(M). Thus $r(a) \leq_T \mathrm{Diag}(M)$. As $\mathbb{R}e(M)$ is a Scott set, by 1.6 $r(a) <_T \mathrm{Diag}(M)$. //

For future use we note that the way of choosing an index for r(a) in Diag(M) is uniform in a. Tennenbaum's theorem is now obvious.

Corollary 1.9: (Tennenbaum [T]). If $M \models PA$ is nonstandard, then $\deg(M) \neq \underset{\sim}{0}$.

We do not need the full force of PA to prove Corollary 1.9. McAloon [Mc] shows that 1.9 still holds if we weaken the induction schema to formulas with bounded quantifiers. Wilmers [W3] has strengthened this by showing we need only the induction schema for formulas which in prenex form have only bounded existential quantifiers. On the other hand Sheperdson [Sh] has shown that if we restrict the induction schema to quantifier free formulas then we can find recursive nonstandard models.

After 1.8 we stated that r(a) is recursive in Diag(M) uniformly in a. Let us make this more precise.

<u>Definition 1.10</u>: Let $S \subseteq P(\omega)$ such that $|S| = \aleph_0$. Let $E \subseteq \omega^2$. E is a d-<u>enumeration</u> of S iff $E \equiv_T d$ and $S = \{E_n : n \in \omega\}$, where $E_n = \{m : (n,m) \in E\}$.

It might seem more natural to only require $E \leq_T d$, but the next lemma shows these notions are equivalent for all interesting S.

<u>Proposition 1.11</u>: If $S \subseteq P(\omega)$, $|S| = \aleph_0$, $\{X \in S : 0 \in X\}$ is infinite and co-infinite $e_T \geq d$ and S has a d-enumeration E, then S has an e-enumeration E^*.

<u>Proof</u>: Find $h:\omega \to \omega$ bijective such that $0 \in E_{h(n)}$ iff $n \in e$ and $h \leq_T d$. Let $(n,m) \in E^*$ iff $(h^{-1}(n),m) \in E$. Clearly E^* is an enumeration of S. $E^* \leq_T e$ and since $n \in e$ if $0 \in E^*_n$, $e \leq_T d$. //

Our notion of enumeration is tied in with the notion of weak uniform upper bound in recursion theory. See Hodes [Ho] for details . Enumerations arise in arithmetic in the following way.

<u>Lemma 1.12:</u> If $M \models PA$ is nonstandard, there is a Diag(M)-enumeration of $\mathrm{Re}(M)$.

<u>Proof</u>: Let $E \subseteq \omega^2$ be $\{(a,n) : M \models p_n | a\}$. The proof of 1.8 and the subsequent remarks show that E is an enumeration of S and $E \leq_T \mathrm{Diag}(M)$. The result follows from 1.11. //

The following notion of Solovay's seems unnatural but is quite usefull.

<u>Definition 1.13</u>: If S is a countable Scott set, a d-<u>effective enumeration</u> of S is a collection $E \subseteq \omega^2$, $f_1:\omega^2 \to \omega$, $f_2:\omega^2 \to \omega$ and $f_3:\omega \to \omega$ such that:

i) $E \equiv_T d$ and f_1, f_2, f_3 are d-recursive.

ii) $S = \{E_n : n \in \omega\}$.

iii) $E_{f_1(m,n)} = E_m \times E_n$.

iv) If $\{e\}^{E_n} \in 2^\omega$ then $E_{f_2(e,n)} = W_e^{E_n}$.

v) If E_n is an infinite tree, $E_{f_3(n)}$ is an infinite path through T.

Surprisingly effective enumerations arise naturally.

Lemma 1.14: (Solovay [So]). If $M \models PA$ is nonstandard, then there is a Diag(M)-effective enumeration of $\mathbb{R}e(M)$.

Proof: Let E be as in the proof of 1.12 (we will eventually have to use 1.11 to pad E, but we ignore this complication). We must define f_1, f_2 and f_3.

1) In Claim 1 of the proof of 1.5 we showed that $M \models \forall u,v \; \exists w \; \exists i < d \; (p_i | w \leftrightarrow (p_{\sigma_0(i)} | u \wedge p_{\sigma_1(i)} | v))$. By Martijasevič's theorem [D-M-R], there is a quantifier free formula $\psi(u,v,w,x,\bar{z})$ such that $PA \models [\forall i \leq x \; (p_i | w \leftrightarrow (p_{\sigma_0(i)} | u \wedge p_{\sigma_1(i)} | v))] \leftrightarrow \exists \bar{z} \; \psi(u,v,w,x,\bar{z})$.

We define f_1 as follows. Given a,b we search for a c and a $\bar{c}$ such that $\psi(a,b,c,d,\bar{c})$. This search can be carried out effectively in Diag(M). We will eventually find such a c and $\bar{c}$. The first time we do we let $f_1(a,b) = c$.

2) In Claim 2 of the proof of 1.5 we showed that $M \models \forall u,v \; \exists w \; \forall i < d \; (p_i | w \leftrightarrow \{v\}^{\{b : p_b | u\}}(i)$ converges in d steps and outputs 1), where this is suitably formalized using the Δ_0 definition of the Kleene T-predicate. As above we can replace the bounded quantifier portion of this by an existential formula. We can effectively search for instantiations of the existential formulas and if $\{e\}^{r(a)} \in 2^\omega$,

the witnesses we find will give a c coding $W_e^{r(a)}$. Let $f_2(e,a) = c$.

3) Given any $u \in M$, $M \models$ "$\exists v, \sigma \in 2^{\underset{\smile}{M}}$ such that σ is in the finite subset of $2^{\underset{\smile}{M}}$ coded by $\{b \in M : p_b | u\}$, and σ is of maximal length such that every initial subsequence of σ is in the set coded by $\{b \in M : p_b | u\}$, and $\forall i < |\sigma|\ p_i | v \leftrightarrow \sigma(i) = 1$." If $r(a)$ is a tree and c witnesses this, then $r(c)$ is an infinite path through T. As in 1) and 2) above we can replace the Δ_0-part of this by an existential formula and effectively search for witnesses. We define $f_3(a)$ to be the least such witness. //

At first d-effective enumerations seem much rarer than d-enumerations, but we will show in §7 that if there is a d-enumeration of S, then there is a d-effective enumeration. From time to time we will tacitly use the following fact.

Proposition 1.15: If there is a d-effective enumeration of S and $e \geq d$, then there is an e-effective enumeration.

Proof: As in 1.11. //

When studying models of arithmetic from a recursion theoretic point of view, we are forced to look for invariants of M which have a recursion theoretic character. One candidate is $\mathbb{R}e(M)$. Another is $Th(M)$, the complete first order theory of M. The remainder of this section is devoted to the relationship between $\mathbb{R}e(M)$, $Th(M)$ and $deg(M)$. Our principle model theoretic tool is the following result implicit in Friedman [Fr].

Lemma 1.16: Let M be a nonstandard model of PA. Let $\Gamma(v,\bar{c})$ be a recursive set of Σ_n-formulas in the free variable v and parameters $\bar{c}$, which is consistent with $Th(M,\bar{c})$. There is an $a \in M$ realizing

$\Gamma(v,\bar{c})$.

Proof: Let Sat_n be the Σ_n-predicate defining satisfaction for Σ_n formulas (i.e., if $\varphi(\bar{v})$ is a Σ_n-formula and $\bar{a} \in M$, $M \models Sat_n(\ulcorner\varphi\urcorner, \ulcorner\bar{a}\urcorner) <-> \varphi(\bar{a})$. Let $\gamma_0, \gamma_1, \gamma_2 \cdots$ be a recursive listing of Γ. Using the fact that recursive sets can be represented in arithmetic, we can find a formula $\Gamma(v,w)$ such that for any $a \in M$, $M \models \forall w(\psi(a,w) <-> Sat_n(\ulcorner \bigwedge_{i=0}^{a} \gamma_i(v) \urcorner, w))$. As $\Gamma(v)$ is consistent, for any $n \in \omega$ $M \models \exists w\, \psi(n,w)$. Thus, by overspill this must also be true for some nonstandard $a \in M$. Let b witness this. Then b realizes $\Gamma(v)$. //

Corollary 1.17: If $M \models PA$ is nonstandard, then for each $n \in \omega$

$$Th(M) \cap \Sigma_n \in Re(M).$$

Proof: Let $\Gamma(v)$ be the type $\{p_\varphi | v <-> \varphi : \varphi \in \Sigma_n\}$. By 1.16 $\Gamma(v)$ is realized in M. //

Corollary 1.18: (Feferman [F]). If $M \models Th(\mathbb{N})$ is nonstandard, then $Re(M)$ contains all arithmetic sets.

Knight showed that 1.17 gives the only restriction on the complete theory of a countable model of PA with a given Scott set.

Theorem 1.19: (Knight [K1]). If S is a countable Scott set and T is a complete extension of PA such that every $T \cap \Sigma_n \in S$, then there is $M \models T$ with $Re(M) = S$.

We will show that for any countable Scott set S, there are arbitrarily complex T such that for any n $T \cap \Sigma_n \in S$. We will need the following lemma used by Harrington [H].

Lemma 1.20: There is a recursive sequence of sentences $\sigma_2, \sigma_3, \sigma_4 \cdots$ such that σ_n is a Δ_n sentence and σ_n is independent of $PA \cup T$, where T is any set of Σ_{n-1} sentences consistent with PA.

Proof: Using Gödel's diagonalization lemma [Sm1], we can find a Rosser sentence σ_n such that

$\sigma_n \leftrightarrow$ "If there is a proof of σ_n from true Σ_{n-1} sentences, then there is a proof with smaller Godel code of $\neg\sigma_n$ from true Σ_{n-1} sentences." Here by proof we always mean proof from PA.

σ_n is easily seen to be Π_n. Let σ_n' be the sentence "Either there is no proof of σ_n from true Σ_{n-1} sentences or there is a proof of $\neg\sigma_n$ from true Σ_{n-1} sentences for which there is no proof of σ_n from true Σ_{n-1} sentences with a smaller Gödel code." The first disjunct of σ_n' is Π_{n-1}, the second is Σ_n. Thus σ_n' is Σ_n. Further, $PA \vdash \sigma_n \leftrightarrow \sigma_n'$, so σ_n is Δ_n.

Suppose $T \subseteq \Sigma_{n-1}$ is consistent with PA and $T \cup PA \vdash \sigma_n$. Then there is a proof of PA from true Σ_{n-1} sentences, so there is a proof of $\neg\sigma_n$. Hence $PA \cup T \vdash \neg\sigma_n$, a contradiction. On the other hand if $T \cup PA \vdash \neg\sigma_n$, then there is a proof of σ_n from true Σ_{n-1} sentences. Hence $T \cup PA \vdash \sigma_n$, a contradiction. Thus σ_n is sufficiently independent. //

Theorem 1.21: If S is a countable Scott set and $d \subseteq \omega$, there is a complete extension $T \supseteq PA$ such that $T \cap \Sigma_n \in S$ for each $n \in \omega$ and $d \leq_T T$.

Proof: We will build T as a union of $T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$ such that for $n \geq 1$ $T \cap \Sigma_n \subseteq T_n \subseteq \Sigma_{2n+2}$, each $T_n \in S$ and T is a complete extension of PA. Let $T_0 = \emptyset$. Let $\sigma_2, \sigma_3, \sigma_4 \cdots$ be as in Lemma 1.20.

<u>Step n</u>: Suppose we have $T_{n-1} \subseteq \Sigma_{2n}$. Let $T^* = T_{n-1} \cup PA \cup \{\varphi\}$. Where φ is σ_{2n+1} if $n \in d$ and $\neg\sigma_{2n+1}$ if $n \notin d$. T^* is consistent and $T^* \in S$. Thus by 1.3 there is a complete $T^{**} \supseteq T$ in S. Let $T_n = T^{**} \cap \Sigma_{2n+2}$. Clearly, $T_n \in S$.

Let $T = \cup T_n$. It is easily seen that each $T \cap \Sigma_n$ is in S and as $n \in d$ iff $\sigma_{2n+1} \in T$, $d \leq_T T$. //

In Theorem 1.27 we will show conditions sufficient to insure there is $T \equiv_T d$ and in Section 7 we will show these conditions are necessary.

<u>Definition 1.22</u>: Let T be a complete extension of PA. We say $X \subseteq \omega$ is <u>represented</u> in T iff there is a parameter free formula $\varphi(v)$ such that for each $n \in \omega$ $n \in X$ iff $\varphi(n) \in T$. Rep(T) is the set of all subsets of ω represented in T.

<u>Lemma 1.23</u>: If T is a complete extension of PA, $T \neq Th(\mathbb{N})$ and M is the minimal model of T, then $Rep(T) = \mathbb{R}e(M)$.

<u>Proof</u>: First note that since PA has definable Skolem functions, there is a pointwise definable minimal model of T. Let $\varphi(v)$ be a parameter free formula. Let $\Gamma(v) = \{p_n | v \leftrightarrow \varphi(n) : n \in \omega\}$. By 1.16 Γ is realized in M by some element a. But then $\varphi(n)$ iff $n \in r(a)$, so $Rep(T) \subseteq \mathbb{R}e(M)$.

Let $a \in M$. As M is minimal there is a formula $\psi(v)$ such that $M \models \psi(a) \wedge \forall v(\psi(v) \rightarrow w = a)$. Let $\varphi(v)$ be the formula $\forall w(\psi(w) \rightarrow p_v | w) \wedge \exists w(\psi(w) \wedge p_v | w)$. Then $p_n | a$ iff $\varphi(n) \in T$. So $r(a) \in Rep(T)$. Thus $Rep\ T = \mathbb{R}e(M)$. //

<u>Corollary 1.24</u>: (Scott-Tennenbaum [Sc-T]). If T is a complete extension of PA, then Rep(T) is a Scott set.

Proof: If $T \neq Th(\mathbb{N})$ this follows from 1.23. If $T = Th(\mathbb{N})$, then $Rep(T)$ is just the arithmetic sets. //

Scott [Sc] proved the converse of 1.24.

Theorem 1.25: (Scott [Sc]). If S is a countable Scott set, there is a complete $T \supseteq PA$ such that $S = Rep(T)$.

We will modify Scott's proof to obtain an effective version of 1.25. We use the following lemma.

Lemma 1.26: For each $n \geq 2$ there is a recursive sequence of Δ_n sentences $\gamma_0^n, \gamma_1^n, \gamma_2^n \ldots$ such that for any set of Σ_{n-1} formulas T, which is consistent with PA, and any $\tau \in 2^\omega$ $PA \cup T \cup \{ \bigwedge_{\tau(i)=1} \gamma_i^n \wedge \bigwedge_{\tau(i)=0} \neg\gamma_i^n\}$ is consistent.

Proof: Using Gödel's diagonalization lemma we can find γ_i^n such that $\gamma_i^n \leftrightarrow$ "For any proof of γ_i^n from PA, true Σ_{n-1} sentences and the true assignment of $\gamma_0^n \ldots \gamma_{i-1}^n$, there is a shorter proof of $\neg\gamma_i^n$ from the same hypothesis." The verification that this sequence works is similar to the verification in Lemma 1.20. We use heavily the inductive assumption that $\gamma_1^n, \ldots, \gamma_{i-1}^n$ are Δ_n. //

Theorem 1.27: Suppose there is a d-effective enumeration of a Scott set S. Then there is a complete $T \supseteq PA$ such that $Rep(T) = S$ and $T \equiv_T d$.

Proof: Let E, f_1, f_2, f_3 be the d-effective enumeration of S. Recursively in d, we build $T = \cup T_n$ where $T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots$, $T_n \in S$ $T \cap \Sigma_n \subseteq T_n \subseteq \Sigma_{2n+2}$ and T is a complete extension of PA. Let $T_0 = \emptyset$.

Step n = 2m: (We proceed as in the proof of 1.21). Let s be such that $E_s = T_{n-1}$. Using f_1 and f_2 we can effectively find an s_0 such that $E_{s_0} = T_{n-1} \cup PA \cup \{\varphi\}$ where φ is σ_{2n+1} if $n \in d$ and $\neg\sigma_{2n+1}$ if $n \notin d$ (here $<\sigma_n : n \in \omega>$ is as in Lemma 1.20), and an s_1 such that E_{s_1} is a tree with every infinite path yielding a completion of E_{s_0}. Using f_3 we can effectively find t so that E_t is a path through E_{s_1}. Finally, using f_1 and f_2 we can find t' such that $E_{t'} = E_t \cap \Sigma_{2n+2}$. Let $T_n = E_{t'}$.

<u>Step $n = 2m+1$</u>: (We try to code E_m into $T \cap \Sigma_{2n+1}$). Let s code T_n. Using f_1, f_2 and f_3 as above we can find a t so that t codes $T^* \cap \Sigma_{2n+2}$, where T^* is a completion of $T_{n-1} \cup PA \cup \{\gamma_i^{2n+1} : i \in E_m\} \cup \{\neg\gamma_i^{2n+1} : i \notin E_m\}$.

As our construction is recursive in d it is clear that $T \equiv_T d$. It also follows that T is a complete extension of PA, $T \cap \Sigma_n \in S$ for each $n \in \omega$, and $E_m \leq_T T \cap \Sigma_{4_m+3}$. Since each $T \cap \Sigma_n \in S$, $\mathrm{Rep}(T) \subseteq S$. But since each $E_m \in S$ is recursive in some $T \cap \Sigma_n$ and $\mathrm{Rep}(T)$ is a Scott set, $S \subseteq \mathrm{Rep}(T)$. Thus $\mathrm{Rep}(T) = S$. //

In Section 7 we will sharpen this result to give a necessary and sufficient condition.

Earlier we wondered what effect our choice of $\mathbb{R}\mathrm{e}(M)$ has in determining $\mathrm{Th}(M)$. The next easy corollary shows it is very small.

<u>Corollary 1.28</u>: Let S be a Scott set. There are distinct $\{M_\alpha : \alpha < 2^{\aleph_0}\}$ such that M_α is minimal and $\mathbb{R}\mathrm{e}(M_\alpha) = S$. In particular, if $\alpha \neq \beta$ $M_\alpha \not\equiv M_\beta$.

In fact, even after we have fixed $\mathbb{R}\mathrm{e}(M)$ and $\mathrm{Th}(M)$ we are a long way from characterizing M.

<u>Proposition 1.29</u>: Let S be a Scott set and let $T \neq Th(\mathbb{N})$ be complete such that $Rep(T) = S$. There are $\{M_\alpha : \alpha < 2^{\aleph_0}\}$ such that for each $\alpha, M_\alpha \models T$, $\mathbb{R}e(M_\alpha) = S$ and $|M_\alpha| = \aleph_0$ but if $\alpha \neq \beta$ $M_\alpha \not\cong M_\beta$.

Proposition 1.29 follows from a deep theorem of Gaifman via a simple observation.

<u>Lemma 1.30</u>: If $M, N \models PA$ and N is an end extension of M then $\mathbb{R}e(M) = \mathbb{R}e(N)$.

<u>Proof</u>: Let $a \in N$ and let $b \in M$. Consider the type $\Gamma(v) = \{p_n | v \Longleftrightarrow p_n | a : n \in \omega\} \cup \{v < b\}$. This is consistent and hence realized by a $c \in M$. Then $r(a) = r(c)$. //

<u>Theorem 1.31</u>: (Gaifman [G]). If $M \models PA$ is minimal, there are $2^{\aleph_0}$ nonisomorphic countable elementary end extensions of M.

In Section 5 we will restrict our considerations to recursively saturated models. Here $\mathbb{R}e(M)$ and $Th(M)$ determine M up to isomorphism.

§2. DEGREES CODING COMPLETE EXTENSIONS OF PA.

In this section we will survey some of the known results on degrees coding models of arithmetic. Tennenbaum's theorem (Corollary 1.9) shows that if M is a nonstandard model of PA, then $\deg(M) \not\leq \underset{\sim}{0}$. More precisely, there is $X \leq_T \mathrm{Diag}\, M$ such that X separates a pair of effectively inseparable r.e. sets. (Recall r.e. sets A and B are <u>effectively inseparable</u> if there is a partial recursive function $\psi(x,y)$ so that if $A \subseteq W_x$, $B \subseteq W_y$ and $W_x \cap W_y = \emptyset$, then $\psi(x,y) \in \omega\ (W_x \cup W_y)$. The sets $A_\emptyset$ and $B_\emptyset$ used in 1.6 are effectively inseparable as are the sets $\{\ulcorner\varphi\urcorner : PA \vdash \varphi\}$ and $\{\ulcorner\varphi\urcorner : PA \vdash \neg\varphi\}$. See Rogers [R] or Smullyan [Smu] for more information.) We have also shown that $\mathrm{Diag}(M)$ enumerates a Scott set. The following theorem gives a complete classification of degrees coding nonstandard models of PA. It appears (with the exception of iv) in Simpson [Si] where it is attributed to a combination of Scott and Tennenbaum [Sc-T], Solovay and Jockusch and Soare [J-S].

<u>Theorem 2.1</u>: The following are equivalent.

i) There is a nonstandard $M \models PA$ such that $\mathrm{Diag}(M) \equiv_T d$.

ii) There is a complete extension T of PA such that $T \equiv_T d$.

iii) There is $X \equiv_T d$ which separates a pair of effectively inseparable r.e. sets.

iv) There is a d effective enumeration of some Scott set S.

Before embarking on the proof of 2.1 we prove a usefull padding lemma.

<u>Lemma 2.2</u>: If $M \models PA$ and $e_T \geq \mathrm{Diag}(M)$, then there is $N \cong M$ with $\mathrm{Diag}(N) \equiv_T e$.

Proof: Let $M = (\omega,\oplus,\odot)$. We can find an e-recursive bijection $f:\omega \to \omega$ such that for any $n \in \omega$ $n \in e$ iff $M \models$ "$f(m)$ is even."

Define $\oplus'$ and $\odot'$ so that

$$m_0 \oplus' m_1 = m_2 \text{ iff } f(m_0) \oplus f(m_1) = f(m_2)$$

and

$$m_0 \odot' m_1 = m_2 \text{ if } f(m_0) \odot f(m_1) = f(m_2).$$

Let $N = (\omega,\oplus',\odot')$. Clearly, $f:N \to M$ is an isomorphism and $\mathrm{Diag}(N) \equiv_T e$. //

We can now prove the equivalence of i), ii) and iv).

i) $\Rightarrow$ iv) This is Solovay's Lemma 1.14.

iv) $\Rightarrow$ ii) This follows from Theorem 1.27.

ii) $\Rightarrow$ i) Given any T by an effective Henkin argument we can find $M \models T$ such that the full diagram ($\mathrm{Diag}^*(M)$) of M is recursive in T. By Lemma 2.2 we can pad M to $N \models T$ with $\mathrm{Diag}(N) \equiv_T T$.

A corollary to this equivalence is the following unpublished result of Solovay.

Corollary 2.3: If T is a complete extension of PA and $e_T \geq T$, there is another complete extension T' where $T' \equiv_T e$.

Proof: This follows from 2.2 and the i) $\Rightarrow$ ii) portion of 2.1. (For the sake of completeness we would like to sketch Solovay's original proof) Let $\varphi_0,\varphi_1,\varphi_2\ldots$ list all sentences in the language of PA. At any stage s we will have formed a sentence ψ_s a conjunction of φ_i or $\neg\varphi_i$ $i \leq s$. Assume at stage $s = 2m$ ψ_{s-1} is consistent with PA. Ask T if $\psi_{s-1} \wedge \varphi_m$ is consistent with PA. If $T \vdash \mathrm{Con}(\ulcorner \psi_{s-1} \wedge \varphi_m \urcorner)$, then ψ_s is consistent with PA as Π^0_1 sentences are preserved under initial segments. If $T \vdash Con(\ulcorner\psi_{s-1} \wedge \varphi_m\urcorner)$, ask if $T \vdash Con(\ulcorner\psi_{s-1} \wedge \neg\varphi_m\urcorner)$. If it does, let $\psi_s = \psi_{s-1} \wedge \neg\varphi_m$. Suppose $T \vdash \neg Con(\ulcorner\psi_{s-1} \wedge \varphi_m\urcorner) \wedge \neg Con(\ulcorner\psi_{s-1} \wedge \neg\varphi_m\urcorner)$. In T we can prove there are x and y such that x and y are the least Godel codes of a proof of $0 = 1$ from $\psi_{s-1} \wedge \varphi_m$ and $\psi_{s-1} \wedge \neg\varphi_m$, respectively. As ψ_s is consistent with PA at most one of x and y may be standard. Thus if we add $\pm\varphi_m$ to ψ_{s-1} to maximize the code of the shortest proof of a contradiction, we maintain consistency. This step can be done recursively in T.

At stage $s = 2m+1$ we use the independent sentences from 1.20 as we did in 1.21 to code e into T. (The entire construction can be done recursively in e.) //

Another corollary to the equivalences we have proved so far is the following corollary which generalizes the Scott-Tennenbaum result that no complete extension of PA has minimal degree. (Jockusch and Soare [J-S] have shown that any countable partial ordering can be embedded in the Turing degrees below a complete extension of PA.)

Corollary 2.4: If T is a complete extension of PA, then there is another complete extension T' such that $T' <_T T$.

Proof: By ii) $\Rightarrow$ iv) there is a Scott set S with a T effective enumeration. But by 1.3 S contains T' a complete extension of PA. By 1.6 $T' <_T T$. //

Continuing the proof of 2.1, we notice that our proof of 1.9 shows i) $\Rightarrow$ iii). We will finish the proof by showing iii) $\Rightarrow$ iv). We use the following theorem of Smullyan.

Theorem 2.5: (Smullyan [Smu]). Suppose (A,B) and (C,D) are two pairs of effectively inseparable r.e. sets. Then there is a recursive bijection $f:\omega \to \omega$ such that $f(A) = C$ and $f(B) = D$. Moreover, given indices for A,B,C,D and the functions witnessing effective inseparability, we can effectively find an index for f.

Proof of iii) $\Rightarrow$ i): Let $\varphi_1, \varphi_2, \varphi_3 \cdots$ list all sentences in the language of PA. Let (A,B) be a pair of effectively separable r.e. sets. Let $\psi_0(x,y)$ witness this. Let $X \equiv_T d$ be such that $A \subseteq X$ and $B \cap X = \emptyset$. Let $P_0 = \{\varphi : PA \vdash \varphi\}$ and $R_0 = \{\varphi : PA \vdash \ \varphi\}$. As noted above P_0 and R_0 are effectively inseparable. Let $\psi_1(x,y)$ witness this.

We build a complete theory recursively in X. At each step s we build θ_s a finite conjunction of $\pm\varphi_i$ $i \le s$, and r.e. sets $P_s = \{\varphi : PA \vdash \theta_s \to \varphi\}$, $R_s = \{\varphi : PA \vdash \theta_s \to \neg\varphi\}$. As long as θ_s is consistent with PA, P_s and R_s are effectively inseparable r.e. sets. Note that $\psi_1(x,y)$ is still a witness for the inseparability of P_s and R_s. Thus by 1.4 we can effectively find a recursive bijection $f:\omega \to \omega$ such that $f(A) = P_s$ and $f(B) = R_s$. Look at φ_{s+1}. Recursively in X we can decide if $\varphi_{s+1} \in f(X)$. If it is, let $\theta_{s+1} = \theta_s \wedge \varphi_{s+1}$. As $f(X)$ separates P_s and R_s, $PA \nvdash \theta_s \to \neg\varphi_{s+1}$, so θ_{s+1} is consistent with PA. If $\varphi_{s+1} \notin f(X)$, let $\theta_{s+1} = \theta_s \wedge \neg\varphi_{s+1}$. As $\theta_{s+1} \notin f(X)$, $PA \nvdash \theta_s \to \varphi_{s+1}$. Thus θ_{s+1} is consistent. Let $T = \{\theta_s : s \in \omega\}$. T is a complete extension of PA and $T \le_D d$. By Corollary 2.3 we can find another complete T^* such that $T^* \equiv_T d$.

This concludes the proof of Theorem 2.1. //

Theorem 2.1 can be extended in several ways.

Definition 2.6: If $M = (\omega, \oplus, \odot) \models PA$, let $\mathrm{Diag}^*(M)$ denote the full diagram of M. Let $\deg^*(M)$ be the Turing degree of $\mathrm{Diag}^*(M)$.

It is easily seen that the ii) $\Rightarrow$ i) proof of 2.1 gives $M \models PA$ with $\mathrm{Diag}^*(M) \equiv_T d$. Further, 2.2 shows that if there is $M \models PA$ with $\mathrm{Diag}^*(M) \equiv_T d$, there is $N \cong M$ with $\mathrm{Diag}(M) \equiv_T d$. Thus we could add to 2.1 the equivalence, v). There is a nonstandard $M \models PA$ with $\mathrm{Diag}^*(M) \equiv_T d$.

We will show in Section 7 that we may also add the equivalence vi). There is a Scott set S with a d-enumeration.

§3. DEGREES OF MODELS OF Th($\mathbb{N}$)

In 2.1 we characterized the degrees coding some nonstandard model of Th($\mathbb{N}$). Suppose we fix T a completion of PA. What degrees code nonstandard models of T? One negative restriction is apparent.

<u>Proposition 3.1</u>: If $M \models PA$, then $\deg(M)^{(\omega)} \geq_T Th(M)$.

<u>Proof</u>: This is clear since $Th_{\Sigma_{n+1}}(M) \leq_T (Th_{\Sigma_n}(M))'$ uniformly in n. //

This bound seems somewhat loose as we know that if M is nonstandard each $Th_{\Sigma_n}(M) \leq_T \deg(M)$. But Harrington [H] showed that it is the best possible.

<u>Theorem</u>: (Harrington [H]). There is a nonstandard $M \models PA$ such that $Diag(M) \leq 0'$, while $Th(M) \equiv_T 0^{(\omega)}$.

On the positive side, if $T \neq Th(\mathbb{N})$ and $e \geq_T T$, then the usual Henkin arguments and Lemma 2.2 guarantee the existence of a nonstandard $M \models T$ with $Diag(M) \leq_T e$. This is still true for Th($\mathbb{N}$) though we must excercise some care to insure that the model produced is nonstandard. Knight showed that this condition is unnecessary for Th($\mathbb{N}$).

<u>Theorem 3.3</u>: (Knight [K2]). There is a nonstandard $M \models Th(\mathbb{N})$ with $Diag(M) <_T 0^{(\omega)}$.

<u>Proof</u>: Let $C = \{c_k : k \in \omega\}$ be a set of Henkin constants. We will build M from the constants C. Our construction of M will be recursive in $0^{(\omega)}$. Let $T_0 = Th(\mathbb{N}) \cup \{c_0 > n : n \in \omega\}$. Note that, by

overspill, a sentence $\varphi(c_0,\ldots c_n)$ is consistent with T_0 iff $\forall y\ \exists x_0 > y\ \exists x_1\ldots\exists x_n\ \ \varphi(x_0,\ldots x_n) \in Th(\mathbb{N})$. Thus we can test consistency with T_0 recursively in $0^{(\omega)}$. We will build consistent theories $T_0 \subseteq T_1 \subseteq T_2 \subseteq\ldots$ such that

i) T_{n+1} is a finite extension of T_n.

ii) $\cup T_n$ is complete and Henkinized.

iii) For each $e \in \omega$ there is a finite set of basic sentences X and an $m \in \omega$ such that $X \subseteq T_m$ and for some $k \in \omega$ either:

a) $\{e\}^X(k) = 1$ and $k \notin 0^{(\omega)}$;

b) $\{e\}^X(k) = 0$ and $k \in 0^{(\omega)}$; or

c) for all $X' \supseteq X$ included in any T_n $\{e\}^{X'}(k)$ does not converge.

Requirement ii) is handled in the usual manner. Suppose at stage n we want to handle requirement iii). Suppose this cannot be done. Then there is a finite set of basic sentences X' such that for any $k \in \omega$, $k \in 0^{(\omega)}$ iff there is a finite set of sentences $X \supseteq X'$ such that $T_n \cup X$ is consistent and $\{e\}^X(k) = 1$. We see from this that there is a formula $\varphi(v)$ in the language of arithmetic such that $k \in 0^{(\omega)}$ iff $N \models \varphi(k)$. But then $0^{(\omega)}$ is arithmetic, a contradiction. Thus we can always find an X and k to satisfy iii). Let $T_{n+1} = T_n \cup X$.

Let M be the Henkin model of $\cup T_n$. Condition iii) insures $0^{(\omega)} \not\leq_T \mathrm{Diag}(M)$. Hence $0^{(\omega)} \;{}_T\!\!> \mathrm{Diag}(M)$. //

Fefferman's result (1.18) shows that if $M \models Th(\mathbb{N})$ is nonstandard, then $\deg(M)$ is greater than every arithmetic degree. Jockusch asked what other restrictions could be placed on $\deg(M)$ and Knight conjectured there were none.

Knight's Conjecture: If for all $n \in \omega$, $d \geq_T 0^{(n)}$, then there is $M \models Th(\mathbb{N})$ such that $Diag(M) \equiv_T d$.

In [K-L-S], Knight-Lachlan-Soare shows that Knight's conjecture is false, by producing a degree $\underset{\sim}{d}$ above all the arithmetic degrees such that if $E \subseteq \omega^2$ and $\underset{\sim}{E} \leq \underset{\sim}{d}$, the $\{E_n : n \in \omega\}$ does not contain all the arithmetic sets. (In the language of recursion theory $\underset{\sim}{d}$ is not a subuniform upperbound for the arithmetic degrees.) By 1.17 this shows there is no nonstandard $M \models Th(\mathbb{N})$ with $deg(M) = \underset{\sim}{d}$. Recently Lerman [L] has shown that there is a degree $\underset{\sim}{d}$ above all the arithmetic degrees such that $\underset{\sim}{d}$ is not a subuniform upperbound for the arithmetic degrees and $\underset{\sim}{d}' \geq \underset{\sim}{0}^{(\omega)}$. (The Knight-Lachlon-Soare example has $\underset{\sim}{d}'' = 0^{(\omega)}$). This refutes a conjecture of the author from [M].

Before Knight's conjecture was refuted, there were several positive results. Before stating them let us recall some facts about the degrees below $\underset{\sim}{0}^{(\omega)}$. For proofs see [E].

Theorem 3.4: i) (Enderton-Putnam). If for all $n \in \omega$ $d \geq 0^{(n)}$, then $\underset{\sim}{d}'' \geq_T \underset{\sim}{0}^{(\omega)}$.

ii) (Sacks). There is a $\underset{\sim}{d}$ such that for all $n \in \omega$ $\underset{\sim}{d} \geq 0^{(n)}$ and $\underset{\sim}{d}'' = \underset{\sim}{0}^{(\omega)}$.

Theorem 3.5: (Marker [M]). If for all $n \in \omega$ $d \geq_T 0^{(n)}$, then there is a nonstandard $M \models Th(\mathbb{N})$ such that $Diag(M) \equiv_T d'$.

Section 4 is devoted to the proof of 3.5. Theorem 3.5 and 3.4 ii) give the following immediate corollary.

Corollary 3.6: There is a nonstandard $M \models Th(\mathbb{N})$ such that $(Diag(M))' \equiv_T 0^{(\omega)}$.

Corollary 3.6 can be thought of as an analog of the Jockusch and Soare result that there are nonstandard $M \models PA$ with $(\mathrm{Diag}(M))' \equiv_T 0'$. Theorem 3.5 and Corollary 3.6 were strengthened by Knight-Lachlan-Soare.

Theorem 3.7: (Knight-Lachlan-Soare [K-L-S]). If for any $n \in \omega$ $d >_T 0^{(n)}$, then there is a nonstandard $M \models \mathrm{Th}(\mathbb{N})$ with $(\mathrm{Diag}(M))' \equiv_T d'$.

Corollary 3.8: There is a nonstandard $M \models \mathrm{Th}(\mathbb{N})$ such that $\mathrm{Diag}(M)'' \equiv_T 0^{(\omega)}$.

These results still shed little light on Jockusch's problem. Solovay provided the final solution.

Theorem 3.9: (Solovay [So]). Let S be a Scott set containing all the arithmetic sets. There is $M \models \mathrm{Th}(\mathbb{N})$ with $\mathbb{R}\mathrm{e}(M) = S$ and $\mathrm{Diag}(M) \equiv_T d$ iff there is a d effective enumeration of S.

This will be simplified in Section 7.

§4. THE PROOF OF THEOREM 3.5

This section is devoted to the proof of Theorem 3.5. This proof uses a construction first used by Harrington in his proof of 3.2. We will actually show that if for every $n \in \omega$ $d \geq_T 0^{(n)}$, then there is a nonstandard $M \models Th(\mathbb{N})$ such that $Diag(M) \leq_T d'$. Theorem 3.5 follows from this using Lemma 2.2.

We will describe a three worker construction which produces a complete, consistent, Henkinized theory T such that M is the canonical model of T. Worker 1 will use oracle d' and produce the Σ_1-diagram of M. Worker 2 will use oracle d'' and build the Σ_2-diagram of M. Worker 3 uses oracle d''' and constructs the full diagram of M. As every arithmetic set is recursive in d, $0^{(\omega)} \leq_T d''$ by 3.4 i). Thus workers 2 and 3 each have access to $Th(\mathbb{N})$.

Let L be the language of Peano Arithmetic. Let $C = \{c_i : i \in \omega\}$ be a set of Henkin constants. For $i = 1,2$, let $\{\varphi_j^i : j \in \omega\}$ be a recursive listing of all Σ_i-sentences in $L(C)$. Let $\{\varphi_j^3 : j \in \omega\}$ be a recursive listing of all $L(C)$ sentences. We denote the set of Boolean combinations of Σ_n-formulas as $b\Sigma_n$. $B_n(S)$ will denote all $b\Sigma_n$ consequences of S allowing no constants from C not already mentioned in S. For $i = 1,2$, let $\{\Gamma_j^i(v) : j \in \omega\}$ list all sets Γ of $b\Sigma_i$ formulas allowing one free variable v and finitely many constants from C such that Γ is r.e. in d.

We will assume, via the recursion theorem, that each worker knows the others' strategies. That is, we actually describe a recursive function g such that if x is an index for the strategy used by worker $n+1$, $g(x)$ will be an index for the strategy used by worker n.

By the recursion theorem there is a natural number e s.t. for any oracle A $W_e^A = W_{g(e)}^A$. We use the strategy given by index e.

By any stage s worker j will have committed himself to a finite conjunction T_s^j chosen from $\{\varphi_i^j, \neg\varphi_i^j : i \leq s\}$. T^j will denote $\{T_s^j : s \in \omega\}$. We will arrange things so that $T^1 \subseteq T^2 \subseteq T^3$, $\mathrm{Th}(\mathbb{N}) \subseteq T^3$ and T^3 is complete, consistent and Henkinized.

As $d^{(n+1)}$ is r.e. in $d^{(n)}$, worker n may enumerate worker $n+1$'s oracle and approximate the actions of worker $n+1$. For $n = 1,2$ let $K_{i,j}^{n+1}$ $i \leq j$ denote worker n's approximation to T_i^{n+1} based on the j^{th} approximation to $d^{(n+1)}$. To insure our computation of $K_{i,j}^{n+1}$ converges we continue enumerating $d^{(n+1)}$ and dovetail computations using better approximations to $d^{(n+1)}$. As worker $n+1$'s computations of T_i^{n+1} $i \leq j$ converge, we will eventually find a convergent computation of $K_{i,j}^{n+1}$ $i \leq j$. In time we will have enumerated enough of $d^{(n+1)}$ to correctly answer all of worker $n+1$'s stage i oracle queries. Thus, for each i there is an s such that, for any $s' \geq s$ $K_{i,s'}^{n+1} = T_i^{n+1}$.

For any s and $i < s$, worker 2 will form a set $U_i^2(s)$ which is r.e. in d. (To be perfectly correct, worker 2 will find $u_i^2(s) \in \omega$ which is an index for $U_i^2(s)$ in d. We will not distinguish between $U_i^2(s)$ and $u_i^2(s)$ unless significant confusion arises.) $U_i^2(s)$ will contain $B_2(\mathrm{Th}(\mathbb{N}) \cup K_{i,s}^3)$. Note that for some $m \in \omega$ $K_{i,s}^3$ is a set of Σ_m-sentences. Thus $B_2(\mathrm{Th}(\mathbb{N}) \cup K_{i,s}^3) = \{\varphi \in b\Sigma_2 :$ all constants in φ are contained in $K_{i,s}^3$ and $\mathrm{Th}_{\Sigma_{m+4}}(\mathbb{N}) \vdash \forall\bar{c}(K_{i,s}^3 \rightarrow \varphi)\}$. As $\mathrm{Th}_{\Sigma_{m+4}}(\mathbb{N}) \leq_T d$, $B_2(\mathrm{Th}(\mathbb{N}) \cup K_{i,s}^3)$ is r.e. in d.

If $\Gamma_i^2(v)$ ends up consistent with the full diagram of M, there will be a large s' and $c \in C$ such that for any $s \geq s'$ $U_i^2(s) \supseteq \Gamma_i^2(c)$.

and worker 2 has oracle d", worker 2 could maintain consistency of $T^1 \cup Th(\mathbb{N}) \cup T^2_s$. (Note that as $T^1 \subseteq \Sigma_1$ and $T^1 \cup Th_{\Sigma_4}(\mathbb{N})$ is consistent, $T^1 \cup Th(\mathbb{N})$ is consistent.) Similarly, worker 3 could insure $Th(\mathbb{N}) \cup T^1 \cup T^2 \cup T^3$ is consistent, while completing T^3.

The difficulty arises in insuring that T^3 is Henkinized. For example, suppose worker 2 writes down $\exists x\ \forall y\ \varphi(x,y,\bar{c})$ where φ is quantifier free. In order to provide a witness for $\forall y\ \varphi(x,y,\bar{c})$ we must insure worker 1 has set aside a constant c_0 such that $\exists y(c_0,y,\bar{c})$ is not in T^1. Similarly, if worker 3 writes down $\exists x\ \forall y\ \exists z\ \varphi(x,y,z,\bar{c})$, worker 2 must set aside a c_0 such that $\exists y\ \forall z\ \varphi(c_0,y,z,\bar{c})$ is not in T^2 and for any c_1 worker 1 must insure $\exists z\ \varphi(c_0,c_1,z,\bar{c})$ is in T^1.

This difficulty is overcome by our approximation procedures. If worker 2 writes down $\exists x\ \forall y\ \varphi(x,y,c)$, there will be a later stage where worker 1 realizes that worker 2 wants a witness to $\forall y\ \varphi(x,y,\bar{c})$. At this point worker 1 will find a c_0 such that none of the workers could have considered c_0 by this point and set c_0 aside as a witness (i.e., worker 1 will not write down $\exists y\ \varphi(c_0,y,\bar{c})$). As worker 2 will realize that worker 1 has done this, worker 2 may at some point write down $\forall y\ \varphi(c_0,y,\bar{c})$. From this point on everyone is committed to this choice.

Providing a witness for a Σ^0_n-sentence $\exists x\psi(x)$ where $n > 2$, is a bit more complex. First, worker 2 must provide a witness for the Σ_2-consequences of $\psi(x)$. Secondly, this witness must also have been provided by worker 1 as a witness to the Σ_1-consequences of $\psi(x)$. To insure this occurs, workers 1 and 2 attempt to partially saturate the model. Namely, if they believe it is consistent with the actions of higher level workers, they will set aside a witness for $\Gamma^j_i(v)$.

We will arrange things so that for each i, there is a t such that if $s \geq t$ $U_i^2(s) = U_i^2(t)$ (in fact, $u_i^2(s) = u_i^2(t)$). U_i^2 will denote $U_i^2(s)$ for such an s).

Worker 1 will calculate $K_{i,s}^2$ approximating T_i^2. He will also calculate $\delta_j(s,t)$, an approximation to $u_j^2(s)$ based on the t^{th} approximation to d''. $\Delta_j(s,t)$ will be the set r.e. in d with index $\delta_j(s,t)$. At some stage t, worker 1 will have enumerated a sufficient portion of d'' to guarantee that for any $t' \geq t$ $\delta_j(s,t') = u_j^2(s)$. As above, whenever possible we suppress $\delta_j(s,t)$ and concentrate on $\Delta_j(s,t)$.

For $i < s$, worker 1 will form a set $U_i^1(s)$ r.e. in d (again, we really find an index $u_i^1(s)$ for $U_i^1(s)$). $U_i^1(s)$ will contain $B_1(Th(\mathbb{N}) \cup K_{i,s}^2 \cup \Delta_1(i,s) \cup \cdots \cup \Delta_{i-1}(i,s))$. Note that, as above, this is r.e. in d and in fact, given indices for $\Delta_1(i,s) \ldots \Delta_{i-1}(i,s)$ and $K_{i,s}^2$, we can compute an index for $B_1(Th(\mathbb{N}) \cup K_{i,s}^2 \cup \Delta_1(i,s) \cup \cdots \cup \Delta_{i-1}(i,s))$ effectively in d'. If $\Gamma_i^1(v)$ is eventually consistent with the full diagram of M, there will be a constant $c \in C$ and a stage s such that if $s' \geq s$, then $U_i^1(s') \supseteq \Gamma_i^1(c)$. Again we will arrange things so that $\lim_s u_i^1(s)$ exists. U_i^1 will denote $\lim_s U_i^1(s)$.

<u>The basic ideas:</u>

Before detailing the construction, we should outline some of the ideas behind it.

There would be no difficulties involved if we only had to maintain the consistency of T^1, T^2 and T^3 while completing T^3. Worker 1 would make sure $Th_{\Sigma_4}(\mathbb{N}) \cup T_s^1$ is consistent for each s. As $T^1 \leq_T d'$, Witnessing $\Gamma_i^j(v)$ is given priority over witnessing $\Gamma_k^j(v)$ for $i < k$, to make sure all guesses settle down. If it is consistent for worker 2 to realize $\Gamma_i^2(v)$, then it must be consistent for worker 1 to realize $B_1(\Gamma_i^2(v))$ so worker 1 will have done so. Thus, worker 2 will be able to choose a witness. Similarly, if it is consistent for worker 3 to write down $\exists x\, \psi(x)$, then it must have been consistent for worker 2 to realize $B_2(\psi(v))$. Hence, worker 2 must have done so.

One final point should be made. As worker 1 does not have access to all of $Th(\mathbb{N})$, he can not uniformly compute the consequences of formulas of arbitrary complexity. For this reason we have the intermediate worker 2, which, while it could maintain consistency with $Th(\mathbb{N})$, restricts itself to producing the Σ_2-theory to keep worker 1 happy. Worker 3 may produce sentences of arbitrary complexity, since worker 2 has the resources to compute their consequences.

The construction:

Worker 1: Stage s.

Worker 1 first enumerates a bit more of d''. As worker 1 knows, worker 2 will be maintaining consistency, worker 1 will enumerate enough of d'' to insure that the computations of $K_{i,s}^2$ and $\Delta_1(i,s)\ldots\Delta_{i-1}(i,s)$ converge for $i \leq s$, and that $K_{i,s}^2 \cup \Delta_1(i,s) \cup\ldots\cup \Delta_{i-1}(i,s)$ is consistent for $i \leq s$. Here, by consistent, we mean consistent with $Th(\mathbb{N})$. At first it may seem that this task is beyond worker 1's abilities as he only has d' as an oracle. But if $X \subseteq b\Sigma_2$ then $X \cup Th(\mathbb{N})$ is consistent iff $X \cup Th_{\Sigma_4}(\mathbb{N})$ is consistent. As $Th_{\Sigma_4}(\mathbb{N}) <_T d$, this can be checked effectively in d'.

If φ_s^1 is quantifier free, worker 1 sets $T_s^1 = T_{s-1}^1 \wedge \varphi_s^1$ or $T_s^1 = T_{s-1}^1 \wedge \neg\varphi_s^1$ to maintain consistency of

$$(*) \qquad Th_{\Sigma_4}(\mathbb{N}) \cup K_{i,s}^2 \cup \Delta_1(i,s) \cup \cdots \cup \Delta_{i-1}(i,s) \cup U_1^1(s-1) \cup \cdots \cup U_{i-1}^1(s-1) \cup T_s^1,$$

for all $i \leq j \leq s$ for the largest possible j. If φ_s^1 is not quantifier free, worker 1 sets $T_s^1 = T_{s-1}^1 \wedge \varphi_s^1$ or $T_s^1 = T_{s-1}^1$ to maintain consistency of (*) for all $i \leq j$ for the largest possible $j \leq s$.

We next define $U_i^1(s)$, $i < s$. Let χ_i^s denote $K_{i,s}^2 \cup \Delta_1(i,s) \cup \cdots \cup \Delta_{i-1}(i,s) \cup Th_{\Sigma_4}(\mathbb{N}) \cup U_1^1(s) \cup \cdots \cup U_{i-1}^1(s) \cup T_i^1$.

<u>Case 1</u>: $i < s-1$ and $T_s^1 \cup \chi_i^s \cup \Gamma_i^1(v) \cup U_i^1(s-1)$ is consistent.

a) If $U_i^1(s-1)$ contains a realization of $\Gamma_i^1(v)$, then the set $U_i^1(s) = B_1(\chi_i^s \cup U_i^1(s-1))$.

b) If $U_i^1(s-1)$ does not realize $\Gamma_i^1(v)$, find a constant c which no worker could not have used by this stage. Let

$$U_i^1(s) = B_1(\chi_i^s \cup U_i^1(s-1) \cup \Gamma_i^1(c)).$$

<u>Case 2</u>: $i < s-1$, $T_s^1 \cup \chi_i^s \cup \Gamma_i^1(v)$ is consistent and $T_s^1 \cup \chi_i^s \cup \Gamma_i^1(v) \cup U_i^1(s-1)$ is inconsistent. Let $U_i^1(s) = B_1(\chi_i^s \cup \Gamma_i^1(c))$, where c is a constant which no worker could have used by this stage.

<u>Case 3</u>: $i < s-1$ and $T_s^1 \cup \chi_i^s \cup \Gamma_i^1(v)$ is inconsistent.

If $T_s^1 \cup \chi_i^s \cup U_i^1(s-1)$ is consistent, let $U_i^1(s) = B_1(\chi_i^s \cup U_i^1(s-1))$. Otherwise let $U_i^1(s) = B_1(\chi_i^s)$.

<u>Case 4</u>: $i = s-1$. (In this case $U_{s-1}^1(s-1)$ is not defined). a) If $T_s^1 \cup \chi_i^s \cup \Gamma_i^1(v)$ is consistent, let c be a new constant and let

$U_i^1(s-1) = B_1(X_i^s \cup \Gamma_i^1(c))$.

b) If $T_s^1 \cup X_i^s \cup \Gamma_i^1(v)$ is inconsistent, let $U_i^1(s) = B_1(X_i^s)$.

This concludes worker 1's construction.

There are several things to notice.

-Our program is to maintain consistency, while, if at all possible, realizing the types $\Gamma_i^1(v)$. We give priority to realizing types of lower index. We also give priority to our earlier witnesses.

-By induction it is easily seen that each $U_i^1(s)$ is r.e. in d.

-Suppose at stage s_0, $U_1^1,\ldots U_{i-1}^1$ have settled down and for $s \geq s_0$, $K_{j,s}^2 = T_j^2$ and $U_k^2(j) = \Delta_k(j,s)$ for all $k < j \leq i$. Then from this stage onward X_i^s is fixed. Since for $s > s_0$ T_s^1 will be chosen to maintain consistency with $U_i^1(s_0)$, once we make the decision to omit $\Gamma_i^1(v)$ or realize it with c, we will never be forced to back down from this choice. Thus for $s \geq s_0 + 1$, $U_i^1(s) = U_i^1(s_0+1)$. Thus $\lim_s U_i^1(s)$ exists.

<u>Worker 2</u>: At stage s worker 2 will be in either the "active" mode or the "waiting" mode.

Worker 2 knows that for each i there is an s such that if $s' \geq s$ $u_i^1(s') = u_i^1(s)$. The condition $\forall s' \geq s$ $u_i^1(s') = u_i^1(s)$ is recursive in d'' so worker 2 may calculate indicies for $U_1^1,\ldots U_{s-1}^1$.

Worker 2 enumerates more of d''' and calculates $K_{i,s}^3$, $i \leq s$. As worker 3 is maintaining consistency, worker 2 may enumerate enough of d''' to insure the consistency of $T^1 \cup U_1^1 \cup\cdots\cup U_{s-1}^1 \cup T_{2-1}^2 \cup K_{i,s}^3 \cup Th(\mathbb{N})$ for $i \leq s$.

<u>Case I</u>: Worker 2 is in the active mode.

Worker 2 considers the next φ_i^2 which has not been attended to and sets $T_s^2 = T_{s-1}^2 \wedge \varphi_i^2$ or $T_s^2 = T_{s-1}^2$ to keep $Th(\mathbb{N}) \cup T^1 \cup T_s^2 \cup U_1^1 \cup \cdots \cup U_{s-1}^1 \cup K_{i,s}^3 \cup U_1^2(s-1) \cup \cdots \cup U_{i-1}^2(s-1)$, $i \leq j$ consistent for as large a $j \leq s$ as is possible.

We must define $U_i^2(s)$ for $i < s$. Let χ_i^s denote $T_i^2 \cup K_{i,s}^3 \cup U_1^2(s) \cup \cdots \cup U_{i-1}^2(s)$. Let ψ_s denote $T^1 \cup U_1^1 \cup \cdots \cup U_{s-1}^1 \cup T_s^2$.

<u>Case 1</u>: $i < s-1$ and $\psi_s \cup \chi_i^s \cup \Gamma_i^2(v) \cup U_i^2(s-1)$ is consistent.

a) If $U_i^2(s-1)$ contains a witness for $\Gamma_i^2(v)$, then set $U_i^2 = B_2(\chi_i^s \cup U_i^2(s-1))$.

b) If $U_i^2(s-1)$ contains no realization of $\Gamma_i^2(v)$, we calculate $B_1(T_{s-1}^2 \cup \Gamma_i^2(v) \cup U_1^2(s) \cup \cdots \cup U_{i-1}^2(s))$. This is r.e. in d and thus is $\Gamma_\ell^1(v)$ for some $\ell \in \omega$. In d" we may calculate ℓ. Worker 2 sets $U_j^2(s) = B_2(\chi_j^s)$ for $i \leq j < s$ and goes into the waiting mode to find a witness for Γ_ℓ^1.

<u>Case 2</u>: $i < s-1$, $\psi_s \cup \chi_i^s \cup \Gamma_i^2(v)$ is consistent while $\psi_s \cup \chi_i^s \cup \Gamma_i^2(v) \cup U_i^2(s-1)$ is inconsistent. In this case we act as in 1b.

<u>Case 3</u>: $\psi_s \cup \chi_i^s \cup \Gamma_i^2(v)$ is inconsistent.

a) if $i < s-1$ and $\psi_s \cup \chi_i^s \cup \Gamma_i^2(s-1)$ is consistent, let $U_i^2(s) = B_2(\chi_i^s \cup U_i^2(s-1))$.

b) otherwise let $U_i^2(s) = B_2(\chi_i^s)$.

<u>Case 4</u>: If $\psi_s \cup \chi_i^s \cup \Gamma_i^2(v)$ is consistent and $i = s-1$, proceed as in case 1b.

<u>Case II</u>: Player 2 is waiting for a witness to $\Gamma_\ell^1(v)$ which was demanded at stage s_0.

If $s \leq \ell$, then set $T_s^2 = T_{s_0}^2$, $U_j^2(s) = U_j^2(s_0)$ for $j < s_0$ and $U_i^2(s) = \emptyset$ for $s_0 \leq i < s$. If $s > \ell$, we look at U_ℓ^1 to see if it contains a witness c for $\Gamma_\ell^1(v)$. If it does, we set $U_j^2(s) = B_1(X_i^{s_0} \cup \Gamma_\ell^2(c))$ for $i \leq j < s$ and return to the active mode. If U_ℓ^1 contains no witness we set $U_j^2(s) = U_j^2(s_0)$ for $j < s_0$ and $U_i^2(s) = \emptyset$ for $s_0 \leq i < s$, and return to the active mode.

This concludes worker 2's construction.

Several observations: -Again we maintain consistency while trying to realize any consistent type.

-If we shift into the waiting mode we will eventually shift out of it.

-As before, once $U_1^2, \ldots U_{i-1}^2$ and $K_{i,s}^3$ settle down, U_i^2 will settle down.

-At stage s worker 2 need only know about $U_1^1, \ldots, U_{s-1}^1$. As these are computed without reference to $U_i^2(s)$ $i < s$ we avoid circularity in our use of the recursion theorem.

-Workers 1 and 2 have guaranteed that for $i = 1,2$ if $\Gamma_j^i(v)$ is consistent with the full diagram of M, then it is realized.

Worker 3:

Worker 3 will be in either the "active" mode, the "waiting" mode or the "passive" mode.

If worker 3 is active or waiting and worker 2 shifts from the active to the waiting mode, worker 3 becomes passive.

When worker 2 becomes active again, worker 3 returns to the state he was previously in.

Case 1: Worker 3 is passive.

Worker 3 sets $T^3_s = T^3_{s-1}$.

Case 2: Worker 3 is active.

Worker 3 considers the first φ^3_i which has not yet been considered and sets $T^2_s = T^2_{s-1} \wedge \varphi^3_i$ or $T^2_s = T^2_{s-1} \wedge \neg\varphi^3_i$ to maintain consistency of $Th(N) \cup T^2 \cup T^3_s \cup U^2_1 \cup \cdots \cup U^2_{s-1}$. If we add $\exists x\, \psi(x)$ we switch to the waiting mode to find a witness for $\psi(x)$.

Case 3: Worker 3 is waiting for a witness to $\psi(x)$, where $\exists x\, \psi(x)$ was added at stage s_0.

Worker 3 calculates ℓ s.t. $\Gamma^2_\ell = B_2(T^3_{s_0} \cup \psi(v))$. If $s \leq \ell$, let $T^3_s = T^3_{s_0}$. If $s > \ell$, by our construction there is a witness c to $\Gamma^2_\ell(v)$ in U^2_ℓ. Let $T^3_s = T^3_{s-1} \wedge \psi(c)$.

This concludes worker 3's construction.

We observe that: -The passive mode insures that worker 3 remains inactive during the period when worker 2 is not paying attention to its actions.

-As worker 2 always leaves the waiting mode, worker 3 will eventually leave the passive mode.

-Worker 3 will always return from the waiting mode to the active mode.

-Player 3 insures that T^3 is complete, consistent and Henkinized.

From T^1 we pass to M, the canonical model of T^3, effectively in d'. Thus $Diag(M) \leq_T d'$. To see that M is nonstandard, we need only observe that $\Gamma(v) = \{v \neq n : n \in \omega\}$ is consistent and thus realized in M. This concludes the proof of Theorem 3.5. //

§5. S-SATURATION AND EFFECTIVE MODEL THEORY

In this section we state the basic results about recursively saturated structures which will be used in the following sections. The preliminary material in this section is due primarily to Wilmers [W1],[W2], and Knight and Nadel [KN1],[KN2]. We also prove the main theorem of recursive model theory that we will be using. Throughout we assume all structures are countable, and are not necessarily models of arithmetic.

Definition 5.1: Suppose $S \subseteq P(\omega)$. M is S-saturated iff

i) every pure type realized in M is recursive in some $s \in S$ and

ii) if $p(\bar{x},\bar{y})$ is a pure type recursive in some $s \in S$, $\bar{m} \in M$ and $p(\bar{x},\bar{m})$ is consistent, then $p(\bar{x},\bar{m})$ is realized in M.

The following lemma summarizes the facts we will be using about S-saturation and Scott sets.

If M is an L-structure $Typ(M)$ denotes the set of (codes of) pure types realized in M.

Lemma 5.2: i) If M is S-saturated, then $Typ(M) \subseteq S$ and $Th(M) \in S$.

ii) M is recursively saturated iff M is S-saturated for some Scott set S.

iii) If S is a countable Scott set and T is a complete theory recursive in some $s \in S$, then T has a countable S-saturated model.

iv) If M is countable and S-saturated, then M is determined up to isomorphism by S and $Th(M)$.

<u>Proof</u>: i) is immediate from our definitions.

ii) Clearly if M is S-saturated, then M is recursively saturated. For the converse let $S = \{d \subseteq \omega$: if $\bar{m} \in M$, $\Gamma(v,\bar{m})$ is consistent with $\mathrm{Th}(M)$ and Γ is recursive in d, then $\Gamma(v,\bar{m})$ is realized in $M\}$. Then M is S-saturated and S is a Scott set. See [KN1] for details.

iii) This is proved by a Henkin argument similar to the one used by Scott [Sc].

iv) Any two S-saturated models of a complete theory are ω-homogeneous and realize the same types. //

For many theories 5.2 iii) may be significantly strengthened.

<u>Definition 5.3</u>: T is effectively perfect if for some $n \in \omega$ there is a recursive map $\varphi: 2^{<\omega} \to$ consistent formulas in n-variables s.t.

i) if $\sigma \subseteq \tau$, then $T \vdash \varphi(\tau) \to \varphi(\sigma)$ and

ii) $\varphi(\sigma^\wedge 0)$ and $\varphi(\sigma^\wedge 1)$ are incompatible.

<u>Examples</u>: i) If T is a complete extension of Pressburger arithmetic in any suitable recursive language, then T is effectively perfect. The function φ may be given by $\varphi(\alpha) = \bigwedge\{p_n | v : \alpha(n) = 1\} \wedge \bigwedge\{p_n \nmid v : \alpha(n) = 0\}$.

ii) Macintyre [Mac1] has shown that, with the possible exception of some bizarre cases not known to exist, all theories of infinite fields which are not algebraically closed are effectively perfect.

iii) Wilmers [W2] and Knight-Nadel [K-N1] show that if T is not atomic, then T is T-effectively perfect.

iv) Knight-Nadel [K-N1] show that if T has an infinite recursive sequence of independent formulas in n free variables, then T is effectively perfect.

Theorem 5.4: ([K-N1],[N],[W2]). If T is effectively perfect and $M \models T$ is recursively saturated, then M is S-saturated for a unique Scott set S.

Proof: Fix n and φ witnessing the perfection T. Let $\bar{m} \in M^n$. We say $\bar{m}$ codes $f \in 2^\omega$ iff $M \models \varphi(\alpha)(\bar{m})$ for all $\alpha \subseteq f$. Let $S = \{X \subseteq \omega:$ some $\bar{m} \in M^n$ codes the characteristic function of $X\}$.

Claim: S is a Scott set and M is S-saturated.

(We will show only that S is closed under Turing reducability. The rest of the proof is similar.) Suppose $\bar{m}$ codes f and $f_1 \leq_T f$. Consider the type $\Gamma(\bar{v}) = \{\varphi(B)(\bar{v}):\beta \subseteq f_1\}$. By 5.2 i) and ii) M must realize $\Gamma(v)$ by some $\bar{n}$. This $\bar{n}$ codes f_1. Lemma 5.2 also implies M is S-saturated.

Suppose M is also S'-saturated. Then $S' \supseteq S$. Let $s' \in S'$. Coding arguments similar to the above show s' is coded by some $\bar{m} \in M^n$. Thus $S' = S$. //

Knight-Nadel show that the conclusion of 5.4 also holds if T has pure types of every degree. Some condition on T is evidently necessary as any countable saturated model is S-saturated for all sufficiently large S.

We devote the remainder of this section to the effective model theory needed to study recursively saturated models of arithmetic. In what follows let L be a fixed recursive language. If $\mathfrak{A} = (\omega,\ldots)$ is an L-structure with universe ω, $\mathrm{Diag}(\mathfrak{A})$ denotes the atomic diagram of, and $\mathrm{Diag}^*(\mathfrak{A})$ denotes the full diagram of $\mathfrak{A}$. We let $\deg(\mathfrak{A})$ and

$\deg^*(\mathfrak{A})$ denote the Turing degrees of $\mathrm{Diag}(\mathfrak{A})$ and $\mathrm{Diag}^*(\mathfrak{A})$ respectively. If $\mathfrak{A}$ is a structure, $\mathrm{Typ}(\mathfrak{A})$ denotes the set of pure types realized in $\mathfrak{A}$.

Lemma 5.5: If $\mathfrak{A} = (\omega, \ldots)$ is an L-structure, then there is a $\mathrm{Diag}^*(\mathfrak{A})$-enumeration of $\mathrm{Typ}(\mathfrak{A})$.

Proof: Let $\sigma: \omega \to \omega^{\underset{\sim}{\omega}}$ and $\tau: \omega \to$ L-formulas be recursive bijections. Define $E \subseteq \omega^2$ by $(n,m) \in E$ iff $\mathfrak{A} \models \varphi(i, \ldots i_k)$ where $\sigma(n) = \langle i, \ldots, i_k \rangle$ and $\tau(m) = \varphi(\bar{v})$. //

Our main goal is to prove the converse of 5.5 for recursively saturated structures. This will require ideas of Goncharov [Go] and Peretyat'kin [P] which are

Definition 5.6: Let S be a set of types. A d-enumeration E of S has the d-effective extension property iff there is a d-recursive $g: \omega^2 \to \omega$ s.t. if $p(x_0, \ldots, x_{n-1}) \in S$, say $p = \{\psi : (m, \ulcorner \psi \urcorner) \in E\}$, and $\varphi(x_0, \ldots x_n)$ is consistent with p, then $\{\psi : (g(m, \ulcorner \varphi \urcorner), \ulcorner \psi \urcorner) \in E\}$ is an extension of p in the variables $x_0, \ldots x_n$ containing $\varphi(x_0, \ldots x_n)$.

Theorem 5.7: (Goncharov [Go] and Peretyat'kin [P]). Let $\mathfrak{A}$ be ω-homogeneous. Let $E \subseteq \omega^2$ be a d-enumeration of $\mathrm{Typ}(\mathfrak{A})$ with the d-effective extension property. Then there is $\mathfrak{B} = (\omega, \ldots)$ s.t. $\mathfrak{B} \cong \mathfrak{A}$ and $\mathrm{Diag}^*(\mathfrak{B}) \leq_T d$.

We will show that the hypothesis of 5.7 holds reasonably often. Fix T a complete L-theory.

Definition 5.8: A set S of types over T is Turing closed iff for any $p \in S$ if q is a type over T and $q \leq_T p$, then $q \in S$.

Let $X \subseteq P(\omega)$ with T coded by some $x \in X$. Let $\mathrm{Typ}(X)$ be the set of complete types which are coded in X. $\mathrm{Typ}^n(X)$ will denote the complete n-types coded in X.

Lemma 5.9: If $\mathrm{Typ}(X)$ is Turing closed and E is an enumeration of X, then $\mathrm{Typ}(X)$ has an enumeration recursive in E.

Proof: For notational simplicity we will consider only $\mathrm{Typ}^1(X)$. We dovetail over ω^2 to decide larger and larger parts of each E_m. We say m is active at stage s of our construction iff

i) m is active at each stage $s' < s$.

ii) $\{n \leq s : n \in E_m\}$ codes a set of formulas in one variable consistent with T, and

iii) there is no formula $\varphi(v)$ s.t. $\ulcorner\varphi(v)\urcorner \leq s$, $\ulcorner\neg\varphi(v)\urcorner \leq s$ and neither $\ulcorner\varphi(v)\urcorner$ or $\ulcorner\neg\varphi(v)\urcorner$ are in E_m.

These conditions are all recursive in T and as T is coded in X, they can be answered recursively in E.

If m is not active at stage s, there is a greatest $t < s$ such that m was active at stage t. Thus $\Gamma = \{\varphi(v) : \ulcorner\varphi\urcorner \leq t$ and $\ulcorner\varphi\urcorner \in E_m\} \cup \{\neg\varphi(v) : \ulcorner\varphi\urcorner \leq t$ and $\ulcorner\varphi\urcorner \notin E_m\}$ codes a consistent set of formulas. We may now uniformly, and effectively in T, complete Γ by a Henkin process.

We define $E^* \subseteq \omega^2$ as follows: $(m,n) \in E^*$ iff

i) m is active at stage n and $(m,n) \in E$, or

ii) the Henkin process puts the formula with Godel code n into the extension of Γ.

Clearly $E^* \leq_T E^*$. If p is a type in one variable coded in X and $\varphi \in p$ if $\ulcorner\varphi\urcorner \in E_m$, then $\varphi \in p$ iff $\ulcorner\varphi\urcorner \in E^*_m$. Also if

$q = \{\varphi : \ulcorner\varphi\urcorner \in E_n\}$, then either i) n is active at every stage s, in which case q is in $\mathrm{Typ}^1(X)$ or ii) n is deactivated at some stage s. In this case $q \leq_T T$ and since $\mathrm{Typ}^1(X)$ is Turing closed, $q \in \mathrm{Typ}^1(X)$. Thus E^* is an enumeration of $\mathrm{Typ}^1(X)$.

Similar proofs work for each $\mathrm{Typ}^n(X)$. Putting these enumerations together gives an enumeration of $\mathrm{Typ}(X)$. //

Lemma 5.10: Let T be a complete theory. There is a uniform recursive operator Ψ such that if p is a type over T in the variables $x_0,\ldots x_{n-1}$ and $\varphi(x_0,\ldots x_n)$ is a formula consistent with p, then $\Psi(p,\varphi)$ is a type in the variables $x_0,\ldots,x_n$ s.t. $\Psi(p,\varphi) \supseteq p$, $\varphi \in \Psi(p,\varphi)$ and $\Psi(p,\varphi) \leq_T p$.

Proof: Ψ is just the usual Henkin procedure for finding a complete extension of p containing φ. //

Theorem 5.11: If $S \subseteq P(\omega)$ is closed under Turing reducibility, E is an enumeration of S and $E \leq_T d$, then $\mathrm{Typ}(S)$ has an enumeration G with the d-effective extension property.

Proof: We inductively define a set I of indices by the following rules:

i) if $n \in \omega$, then $n \in I$, and

ii) if $i \in I$ and $\varphi(x_0,\ldots x_n)$ is a formula, then $\langle i,\varphi\rangle \in I$.

Let $E^* \subseteq \omega^2$ be the enumeration of $\mathrm{Typ}(X)$ given by 5.9. Let $\sigma:\omega \to I$ be a recursive bijection. We define $G \subseteq \omega^2$ inductively on indices as follows:

i) if $\sigma(m) \in \omega$, $G_m = E^*_m$,

ii) if $\sigma(m) = \langle i,\varphi(x_0,\ldots x_n)\rangle$, $\sigma(m_0) = i$ and $q = \{\theta : \ulcorner\theta\urcorner \in G_{m_0}\}$

is a type in the variables $x_0,\ldots x_{n-1}$ consistent with $\varphi(x_0,\ldots,x_n)$, then G_m codes $\Psi(q,\varphi(x_0,\ldots x_n))$,

iii) otherwise $G_m = E_0^*$.

Clearly G is an enumeration of $\mathrm{Typ}(X)$. Let $h:\omega^2 \to \omega$ be defined by $h(m,\ulcorner\varphi\urcorner) = \sigma^{-1}(\langle\sigma(m),\varphi\rangle)$. h witnesses the d-effective extension property. //

We may now state our partial converse to 5.5.

<u>Theorem 5.12</u>: If $\mathfrak{A}$ is ω-homogeneous, S is the set of degrees of pure types realized in $\mathfrak{A}$, $\mathrm{Typ}(\mathfrak{A})$ is Turing closed and E is an enumeration of S, then there is $\mathfrak{B} \cong \mathfrak{A}$ with $\mathrm{Diag}^*(\mathfrak{B}) \leq_T E$.

<u>Proof</u>: Clear from 5.7 and 5.11. //

Goncharov [Go] and Peretyat'kin [P] have examples which show that we cannot omit the assumption that $\mathrm{Typ}(M)$ is Turing closed. We will be using only the following corollary:

<u>Corollary 5.13</u>: If $\mathfrak{A}$ is S-saturated and E is an enumeration of S, then there is $\mathfrak{B} \cong \mathfrak{A}$ with $\mathrm{Diag}^*(\mathfrak{B}) \leq_T E$.

§6. APPLICATIONS TO ARITHMETIC

In this section we will give applications of 5.13 to the study of structures arising in arithmetic. We begin by fixing some notation. If L is a recursive language and $\mathfrak{A} = (\omega,\ldots)$ is an L-structure, then $\mathrm{Diag}(\mathfrak{A})$ denotes atomic diagram of $\mathfrak{A}$ and $\deg(\mathfrak{A})$ is the Turing degree of $\mathrm{Diag}(\mathfrak{A})$, $D(\mathfrak{A}) = \{\deg(\mathfrak{B}) : \mathfrak{A} \cong \mathfrak{B}\}$, and $D^*(\mathfrak{A}) = \{\deg^*(\mathfrak{B}) : \mathfrak{A} \cong \mathfrak{B}\}$. If $S \subseteq P(\omega)$ is countable, $D(S) = \{d:$ there is a d-enumeration of $S\}$.

a) Pressburger arithmetic.

Let T be a complete theory in a language extending $L = \{+,0,1\}$ such that $T \supseteq \mathrm{Th}(\omega,+,0,1)$. For $M \models T$, M_+ denotes the reduct of M to L. If $a \in M$, we define the real coded by a $r(a) = \{n \in \omega: M \models p_n | a\}$ and let $\mathrm{Re}(M) = \{r(a) : a \in M\}$. We state several facts about $\mathrm{Re}(M)$ which were proved in §1 for $M \models PA$.

<u>Lemma 6.1</u>: If $a \in M$, $r(a) \leq_T \deg(M)$. Moreover, there is a $\mathrm{Diag}(M)$-enumeration of $\mathrm{Re}(M)$.

<u>Proof</u>: The proofs of 1.8 and 1.11 work for any $M \models \mathrm{Th}(\omega,+)$. //

We recall from Section 5 that T is effectively perfect. Thus by 5.4, if $M \models T$ is recursively saturated, then M is S-saturated for a unique Scott set S. We will often refer to such an S as <u>the</u> Scott set of M.

<u>Lemma 6.2</u>: If $M \models T$ is recursively saturated and S is the Scott set of M, then $S = \mathrm{Re}(M)$.

Proof: We need only show M is $\mathbb{R}\mathrm{e}(M)$-saturated. Let $\bar{a} \in M$ and let $\Gamma(v) = \{\mathrm{pr}_{\ulcorner\varphi\urcorner} | v \leftrightarrow \varphi(\bar{a}) : \varphi(\bar{v})$ on L-formula$\}$. As M_+ is recursively saturated, Γ is realized by some element of M. Thus every pure type realized is recursive in some element of $\mathbb{R}\mathrm{e}(M)$. Let $\Gamma(\bar{v},\bar{w})$ be a type coded in $\mathbb{R}\mathrm{e}(M)$, i.e., there is $a \in M$ s.t. $\varphi(\bar{v},\bar{w}) \in \Gamma$ iff $\mathrm{pr}_{\ulcorner\varphi\urcorner}|a$. Then for any $\bar{m} \in M$, M must realize $\{\varphi(\bar{v},\bar{m}) \leftrightarrow \mathrm{pr}_{\ulcorner\varphi\urcorner}|a : \varphi$ on L-formula$\}$. Hence M is $\mathbb{R}\mathrm{e}(M)$-saturated. //

Lemma 6.3: If $M \models T$, $\underset{\sim}{d} \in D(M)$, and $\underset{\sim}{e} \geq \underset{\sim}{d}$, then $\underset{\sim}{e} \in D(M)$. Similarly, if $\underset{\sim}{d} \in D^*(M)$, $\underset{\sim}{e} \in D^*(M)$.

Proof: The proof of 2.2 works for $M \models T$. //

Corollary 6.4: If $M \models T$ is S-saturated, then $D(M) = D^*(M) = D(S)$.

Proof: Corollary 5.13 shows $D^*(M)$ is dense in $D(S)$. Clearly $D(M)$ is dense in $D^*(M)$. As $S = \mathrm{Re}(M)$, $D(S)$ is dense in $D^*(M)$. Lemmas 6.3 and 1.12 show that $D(M) = D^*(M) = D(S)$. //

We note that the conclusion $D(M) = D^*(M)$ uses both recursive saturation and the coding power of arithmetic. In general, if we know $\underset{\sim}{d} \in D(M)$ we can only conclude $\underset{\sim}{d}^{(\omega)} \in D^*(M)$. Harrington's theorem (3.2) shows that this is all we can expect. The following proposition gives an example showing that $D(\mathfrak{A}) = D^*(\mathfrak{A})$ is not a general property of recursively saturated models.

Proposition 6.5: There is a complete undecidable ω_1-categorical theory T with a recursively presented ω-saturated model $\mathfrak{A}$. (Thus $\underset{\sim}{0} \in D(\mathfrak{A})$, but $\underset{\sim}{0} \notin D^*(\mathfrak{A})$.)

Proof: If $\{n\}(n)\uparrow$, let $u_n = 2s$ where $s\{n\}(n)$ halts exactly at step s. For all n let

$$U_n = \begin{cases} \{u_n\} & \text{if } \{n\}(n)\uparrow \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\mathfrak{A} = (\omega, U_n : n \in \omega)$ and let $L = \{U_i : i \in \omega\}$. $\mathrm{Th}(\mathfrak{A})$ is undecidable as $\{n\}(n)\uparrow$ iff $\exists v\, U_n(v) \in \mathrm{Th}(\mathfrak{A})$. The predicate "$m \in U_n$" is recursive uniformly in m and n so $\mathfrak{A}$ is recursively presented.

Suppose $\mathfrak{B} \models \mathrm{Th}(\mathfrak{A})$. Let $X^{\mathfrak{B}} = \{b \in \mathfrak{B} : \exists n \in \omega \;\; \mathfrak{B} \models U_n(b)\}$. If $\mathfrak{C} \models \mathrm{Th}(\mathfrak{A})$ $X^{\mathfrak{B}} \cong X^{\mathfrak{C}}$. Thus the isomorphism type of $\mathfrak{B}$ is determined by $\mathfrak{B}\backslash X^{\mathfrak{B}}$. As no relations hold on $\mathfrak{B}\backslash X^{\mathfrak{B}}$, the isomorphism type of $\mathfrak{B}$ is determined by $|\mathfrak{B}\backslash X^{\mathfrak{B}}|$. As $|\mathfrak{A}\backslash X^{\mathfrak{A}}| = \aleph_0$, $\mathfrak{A}$ is ω-saturated. //

b) Skolem arithmetic.

Let T be a complete extension of $\mathrm{Th}(\omega,\cdot)$ is some recursive language extending $\{\cdot,0,1\}$. Let $M \models T$ and let $b \in M$ be an infinite power of 2. If $a \in M$ and $a|b$ we define $r_b^*(a) = \{n : \exists x\; x^{p_n} = a\}$. Let $\mathrm{Re}_b^*(M)$ denote $\{r_b^*(a) : a|b\}$.

Lemma 6.6: If $\underset{\sim}{d} \in D(M)$, then for all $a|b$, $r_b^*(a) \leq_T d$, uniformly in a,b. Further, there is a d-enumeration of $\mathrm{Re}_b^*(M)$.

Proof: Clearly $r_b^*(a)$ is r.e. in d, so we need only show $\{n : \nexists x\; x^{p_n} = a\}$ is r.e. in d. As a is a power of 2 there is an m s.t. $a = 2^m$. Thus $\nexists x\; x^{p_n} = a$ iff $p_n \nmid m$. Thus $\nexists x\; x^{p_n} = a$ iff $\exists x \bigvee_{i<p_n} x^{p_n}\cdot 2^i = a$. Hence, $\{n : \nexists x\; x^{p_n} = a\}$ is r.e. in d. To obtain a d enumeration of $\mathrm{Re}_b^*(M)$ we begin a list $a_0, a_1, a_2 \ldots$ of all elements of M which divide b. Our enumeration is $E = \{(n,m) : m \in r_b^*(a_n)\}$. //

Lemma 6.7: T is effectively rich and if $M \models T$ is S-saturated then $S = \mathrm{Re}_b^*(M)$, where b is any nonstandard power of 2.

Proof: The function $\varphi(\alpha) = \bigwedge\!\!\!\bigwedge \{\exists x\, x^{p_n} = v : \alpha(n) = 1\} \wedge \bigwedge\!\!\!\bigwedge \{\nexists x\, x^{p_n} = v : \alpha(n) = 0\}$ shows that T is effectively perfect. The second part of the proof is done as in 6.2. //

In particular, for recursively saturated $M \models T$, $\mathrm{Re}_b^*(M)$ does not depend on our choice of b, so we may denote S as $\mathrm{Re}^*(M)$.

Corollary 6.8: If $M \models T$ is S-saturated and $\underset{\sim}{d} \in D(M)$, then $\underset{\sim}{d} \in D(S)$.

Proof: Let b be an infinite power of 2. Let $m_0, m_1, m_2 \ldots$ be a listing of M. We begin dovetailing searches for x such that $xm_i = b$ and list as $n_0, n_1, \ldots$ all m_i for which such an x is found. This can be done d-effectively. Let $E = \{(i,j) : j \in r_b^*(n_i)\}$. $E \leq_T d$, since the decision $j \in r_b^*(n_i)$ is d-recursive uniformly in i. By 6.7 E is an enumeration of S. By 1.12 $\underset{\sim}{d} \in D(S)$. //

In order to classify degrees coding recursively saturated models of T we need a padding lemma analogous to 6.3. This was noticed by Julia Knight.

Lemma 6.9: Let $M \models Th(\omega, \cdot)$. Suppose $a, b \in M$ s.t. a is an infinite power of 2 and b is an infinite power of 3. Let $\underset{\sim}{d} \in D(M)$, if $e \;_T\!\geq d$, then $\underset{\sim}{e} \in D(M)$.

Proof: Let $M = (\omega, \Theta)$ with $\Theta \leq_T d$. Pick a and b out of M. We can find a function $f : \omega \to \omega$ a bijection such that

i) $f \leq_T e$,

ii) if $n \in e$, $M \models f(2n) | a$,

iii) if $n \notin e$, $M \models f(2n) | b$.

Define $\boxdot$ by $n \boxdot m = p$ iff $f(n) \boxdot f(m) = f(p)$. $M \cong (\omega, \boxdot)$, $\boxdot \leq_T e$ and as $n \in e$ iff $\exists m\ m \boxdot 2n = a$ and $n \notin e$ iff $\exists m\ m \boxdot 2n = b$, $e \leq_T \boxdot$. //

Knight has shown that we do not really need a and b to prove 6.9, but as we are only considering recursively saturated structures, this assumption is harmless. We may now prove the desired classification result.

<u>Corollary 6.10</u>: If $M \models T$ is S-saturated, then $D(M) = D^*(M) = D(S)$.

<u>Proof</u>: As in 6.4. //

c) Subsystems of PA.

$I\Sigma_n$ will denote the subsystem of PA obtained by restricting the induction schema to Σ_n formulas. EXP will denote $I\Sigma_0 + \forall x, y\ \exists z\ x^y = z$, where this is suitably formalized. It can be shown that $I\Sigma_1 \supsetneqq \text{EXP} \supsetneqq I\Sigma_0 \supsetneqq (\text{Th}(\omega,+) \cup \text{Th}(\omega,\cdot))$.

If M is a $\{+,\cdot,0,1\}$, M_+ will denote the reduct of M to $\{+,0,1\}$ and $M_\cdot$ will denote the reduct of M to $\{\cdot,0,1\}$. Putting together 6.2, 6.4, 6.7, and 6.10, we get the following corollary.

<u>Corollary 6.11</u>: If $M \models I\Sigma_0$ is S-saturated, then $S = \mathbb{R}\text{e}(M) = \mathbb{R}\text{e}^*(M)$ and $D(M) = D^*(M) = D(S) = D(M_+) = D^*(M_+) = D(M_\cdot) = D^*(M_\cdot)$.

Some of our results may be extended to models which are not recursively saturated. We will require the following results of Cegielski-McAloon and Wilmers. ~~theorem~~.

<u>Theorem 6.12</u>: (Cegielski-McAloon-Wilmers [C-M-W]).

i) Let $M \models I\Sigma_0$, then M_+ is recursively saturated.

ii) Let $M \models EXP$, then M_0 is recursively saturated.

Theorem 6.14 extends results of Lipshitz and Nadel [L-N] for PA. Cegielski, McAloon and Wilmers point out that EXP cannot be weakened to $I\Sigma_0$ in ii).

<u>Lemma 6.13</u>: Let $M \models I\Sigma_0$ be nonstandard and let $b \in M$ be an infinite power of 2. Then $\mathbb{R}e(M) = \mathbb{R}e^*(M)$.

<u>Proof</u>: Let $r(a) \in \mathbb{R}e(M)$. For $n \in \omega$ $M \models \exists v \leq a \ \forall m < n(p_m | v \leftrightarrow (\exists z < v \ z^{p_m} = v))$. By Σ_0-overspill this must also hold for some nonstandard N. If c witnesses this, then $r(a) = r^*(c)$. Thus, $\mathbb{R}e(M) \subseteq \mathbb{R}e^*(M)$. The proof of the reverse inclusion is similar. //

<u>Corollary 6.14</u>: If $M \models I\Sigma_0$ is nonstandard, then $D(M_+) = D^*(M_+) = D(\mathbb{R}e(M))$.

<u>Proof</u>: Clear from 6.12, 6.13, and 6.4. //

<u>Corollary 6.15</u>: If $M \models EXP$ is nonstandard, then $D(M_\cdot) = D^*(M_\cdot) = D(\mathbb{R}e(M))$.

<u>Proof</u>: Clear from 6.12, 6.13 and 6.11. //

d) Initial segments.

Let $M \models I\Sigma_0$. Let a be a nonstandard element of M. We define a structure $[0,a]$ with universe $\{x \in M : x \leq a\}$ and functions $+$ and $\cdot$ defined by:

$$x \dot{+} y = \begin{cases} x + y & \text{if } M \models x + y \leq a \\ a & \text{if } M \models x + y > a \end{cases}$$

$$x \dot{\cdot} y = \begin{cases} x \cdot y & \text{if } M \models x \cdot y \leq a \\ a & \text{if } M \models x \cdot y > a. \end{cases}$$

If $b \in [0,a]$, we define $r_*(b) = \{n : p_n | b\}$.

Lemma 6.16: $\mathbb{R}e(M) = \{r_*(b) : b \leq a\}$.

Proof: This follows from the fact that reals are coded by arbitrarily small nonstandard numbers, which we proved in 1.31. //

Corollary 6.17: $D([0,a]) \subseteq D(\mathbb{R}e(M))$.

Lemma 6.18: $D([0,a])$ and $D^*([0,a])$ are closed upward in the Turing degrees.

Proof: Similar to 2.2. //

Corollary 6.19: If $M \models I\Sigma_0$ and $[0,a]$ is $\mathbb{R}e(M)$-saturated, then $D([0,a]) = D^*([0,a]) = D(\mathbb{R}e(M))$.

The following result shows the hypothesis of 6.19 holds reasonably often.

Lemma 6.20: (Lessan [Le]). If $M \models EXP$ and $a \in M$ is nonstandard, then $[0,a]$ is recursively saturated.

We will show in the appendix that the hypothesis of 6.20 can not be weakened to $M \models I\Sigma_0$.

Corollary 6.21: If $M \models EXP$ and $a,b \in M$ are nonstandard then $D([0,a]) = D([0,b]) = D(\mathbb{R}e(M))$.

Proof: Clear from 6.19 and 6.20. //

Stronger results are possible if we look at $[0,a]_+$, the reduct of $[0,a]$ to addition. Clearly, $\mathbb{R}e([0,a]) = \mathbb{R}e([0,a]_+)$.

Lemma 6.22: (Cegielski-McAloon-Wilmers [C-M-W]). If $M \models I\Sigma_0$, then $[0,a]_+$ is recursively saturated.

Corollary 6.23: If $M \models I\Sigma_0$ and $a,b \in M$ are nonstandard, then $D([0,a]_+) = D([0,b]_+) = D(\mathbb{R}e(M))$.

e) Residue Fields.

Suppose $M \models PA$ and $p \in M$ is an infinite prime. Then M/p is a field of characteristic 0.

Lemma 6.24: (Macintyre [Mac2]).

1) M/p is a pseudofinite field (in the sense of Ax[A]), M/p is recursively saturated and every recursively saturated pseudofinite field is of the form M/p.

2) If T is a complete theory of pseudofinite fields, then T is effectively perfect.

Proof: 1) See Macintyre [Mac2].

2) Let φ be the map $\varphi(\alpha) = \bigwedge\{\exists x(v+n = x^2):\alpha(n) = 1\} \wedge \bigwedge\{\neg\exists x(v+n = x^2):\alpha(n) = 0\}$. Macintyre [Mac1] shows that φ witnesses the perfection of T. //

Let T be a complete theory of pseudofinite fields of characteristic 0. Suppose $K \models T$. If $\delta \in K$ is a nonsquare then $\nexists x\ v+n = x^2$ iff $\exists x\ (v+n = \delta\cdot x^2)$. Suppose $a \in K$. Let $r_K(a) = \{n \in \omega:\exists x(a+n = x^2)\}$.

The above remark shows that $r_K(a)$ is recursive in $\deg(K)$ uniformly in a. Let $\mathbb{R}e(K) = \{r_K(a):a \in K\}$. The following lemma is then clear.

Lemma 6.25: 1) There is a $\deg(K)$ enumeration of $\mathbb{R}e(K)$.

2) If K is S-saturated, then $S = \mathbb{R}e(K)$.

Lemma 6.26: Suppose $K = M/p$ and K is S-saturated. Then $S = \mathbb{R}e(M)$.

Proof: As M/p is definable in M, $\mathbb{R}e(K) \subseteq \mathbb{R}e(M)$. Thus by 6.25 $S \subseteq \mathbb{R}e(M)$.

Suppose $a \in M$. Consider the type $\{\exists x\ v+n = x^2 (\text{mod } p) \iff p_n | a : n \in \omega\}$. This is recursive and of bounded complexity. Thus, by 1.16, it is realized in M. Thus $r(a) \in \mathbb{R}e(M/p)$. So $S = \mathbb{R}e(M/p) \supseteq \mathbb{R}e(M)$. //

As usual we must prove a padding lemma for pseudofinite fields.

Lemma 6.27: If $\underset{\sim}{d} \in D(K)$ and $e \geq_T d$, then $\underset{\sim}{e} \in D(K)$. (Similarly $D^*(K)$ is closed upwards.)

Proof: Let $K = (\omega,\oplus_p,\odot_p)$ have degree d. We can find $f:\omega \to \omega$ a bijection such that f is recursive in e and $f(n)$ is a square iff $n \in e$. Let K' be the field induced by f. Then $K' \cong K$ and $\deg(K') = \underset{\sim}{e}$. //

Corollary 6.28: Let $M \models PA$ and let $p \in M$ be a nonstandard prime, then $D(M/p) = D^*(M/p) = D(\mathbb{R}e(M))$. In particular, if b is also a nonstandard prime, $D([0,a]) = D([0,b])$.

Proof: As in 6.4. //

Let $M/p = (\omega,\oplus_p,\odot_p)$. $(\omega,\oplus_p)$ is a divisible abelian group. In fact, $(\omega,\oplus_p) \cong Q^{(\omega)}$. It follows from the next proposition that

$D((\omega,\oplus_p))$ is the set of all Turing degrees.

Proposition 6.29: $\mathbb{Q}^{(\omega)}$ has presentations of every degree.

Proof: Let b be a nonzero element of $\mathbb{Q}^{(\omega)}$ and write $\mathbb{Q}^{(\omega)} = \langle b\rangle \oplus D$, where $\langle b\rangle$ is the span of b and $D \cong \mathbb{Q}^{(\omega)}$. D has a recursive presentation $(\omega,\boxplus)$. Let $g:D \to \omega$ be a recursive isomorphism. Any $x \in \mathbb{Q}^{(\omega)}$ has a unique representation $x = rb + d$, where $d \in D$ and $r \in \mathbb{Q}$.

Let $h:Q \to \omega$ be a bijection. We define $f:\mathbb{Q}^{(\omega)} \to \omega$ as follows. $f(rb+d) = \langle h(r),g(d)\rangle$, where $\langle,\rangle$ is some fixed recursive pairing function. For $i = 0,1,2$ let $f(r_i b = d_i) = n_i$. Then $\mathbb{Q}^{(\omega)} \models n_0 + n_1 = n_2$ iff $d_0 + d_1 = d_2$ and $r_0 + r_1 = r_2$. Thus if $(\omega,\oplus)$ is the group induced by f, $\oplus$ is recursive in h.

On the other hand,

$$h(\tfrac{m}{n}) = k \Leftrightarrow (\omega,\oplus) \models \tfrac{m}{n}f(b) = \langle k,g(0)\rangle$$

$$\Leftrightarrow (\omega,\oplus) \models mf(b) = n\,\langle k,g(0)\rangle.$$

Thus, given the diagram of $(\omega,\oplus)$ we can compute h. As h may be chosen of arbitrary complexity, $\mathbb{Q}^{(\omega)}$ has presentations in every degree. //

The situation for $(\omega,\odot_p)$ is more delicate. Macintyre has shown that in any $M \models PA$ there are nonstandard primes s.t. $D(\omega,\odot_p)$ is the set of all Turing degrees. On the other hand there are primes such that $D(\omega,\odot_p)$ is not the set of all Turing degrees. Macintyre's precise result is that $D(\omega,\odot_p) = \{d{:}d \geq \text{the degree of } Th(\odot_p)\}$.

The results of this section are true if instead of having $M \models PA$ we have $M \models I\Sigma_1$. It is reasonable to conjecture that EXP is enough. As for $I\Sigma_0$, it is not even known that M/p cannot be algebraically closed!

§7. CLASSIFICATION RESULTS

Let T be a complete extension of PA. In this section we classify degrees coding structures related to models of T. We also discuss the relationship between enumerations of Scott sets and effectiver enumerations of Scott sets. The easiest result is the following.

Proposition 7.1: d codes a recursively saturated model of T iff d enumerates a Scott set containing T.

Proof: Clear from 5.2 i and 5.13. //

Corollary 7.2: d codes a recursively saturated model of T iff d enumerates a Scott set containing $0^{(\omega)}$.

Our main tool is the following theorem of Knight [K1].

Theorem 7.3: (Knight [K1]). Let S be a countable Scott set. S occurs as $\mathbb{R}\mathrm{e}(M)$ for some $M \models T$ iff for all $n \in \omega$, $T \cap \Sigma_n \in S$.

Corollary 7.4: (Knight [K1]). $M \models T(\omega,+)$ is expandable to a model of T iff M is recursively saturated and for all $n \in \omega$, $T \cap \Sigma_n \in \mathbb{R}\mathrm{e}(M)$.

Proof: Let $N \models T$ with $\mathbb{R}\mathrm{e}(N) = \mathbb{R}\mathrm{e}(M)$. Then N_+ is $\mathbb{R}\mathrm{e}(M)$-saturated, so $N_+ \cong M$ by 5.2 iv). //

Corollary 7.5: d codes the additive structure of a model of T iff d enumerates a Scott set S containing all $T \cap \Sigma_n$.

Proof: $(\Rightarrow)$ is clear.

$(\Leftarrow)$ Let d enumerate S. By 5.13 we can find $N \models \mathrm{Th}(\omega,+)$ with $\mathbb{R}\mathrm{e}(N) = S$ such that N is recursively saturated and $\deg(N) = \underset{\sim}{d}$.

But if $M \models T$ with $\mathbb{R}e(M) = S$, then $M_+ \cong N$. Thus d codes the additive structure of a model of T. //

Similar results hold for $Th(\omega, \cdot)$.

<u>Corollary 7.6</u>: i) Let $M \models Th(\omega, \cdot)$. M is expandable to a model of T iff M is recursively saturated and for each $n \in \omega$, $T \cap \Sigma_n \in \mathbb{R}e(M)$.

ii) d codes the additive structure of a model of T iff d enumerates a Scott set S containing all $T \cap \Sigma_n$.

<u>Proof</u>: As in 7.4 and 7.5. //

Using the following result of Friedman's we can classify degrees coding initial segments and residue fields.

<u>Theorem 7.7:</u> Let M and N be countable models of PA. M can be embedded as an initial segment of N iff $\mathbb{R}e(M) = \mathbb{R}e(N)$ and $Th_{\Sigma_1}(M) \subseteq Th_{\Sigma_1}(N)$.

<u>Proof</u>: See [Sm2]. //

<u>Corollary 7.8</u>: d codes $[0,a]$ where $a \in M$ is a nonstandard model of T iff d enumerates a Scott set containing each $T \cap \Sigma_n$.

<u>Proof</u>: ($\Rightarrow$) Clear.

($\Leftarrow$) Let d enumerate S. Let $T^* \in S$ be a completion of $PA \cup (T \cap \Sigma_1^0)$ and let $N \models T^*$ be recursively saturated with $\deg(N) = \underset{\sim}{d}$. By 6.7 there is $M \supseteq_e N$ such that $M \models T$. Thus, there is $a \in M$ such that $\deg([0,a]) \leq \underset{\sim}{d}$. By 6.18, $\underset{\sim}{d} \in D([0,a])$. //

<u>Corollary 7.9</u>: d codes M/p where $M \models T$ and $p \in M$ is a nonstandard prime iff d enumerates a Scott set containing all $T \cap \Sigma_n$.

Proof: As in 7.8. //

The next proposition is the surprising result that effectively enumerating a Scott set is no stronger than enumerating it.

Proposition 7.10: If d enumerates a Scott set S, then d effectively enumerates S.

Proof: Let $T \in S$ be a complete extension of PA. By 5.13, there is a recursively saturated $N \models T$ such that $\mathbb{R}\mathrm{e}(N) = S$ and $\deg(N) = d$. By 1.14, there is a d effective enumeration of S. //

7.10 has several corollaries which were hinted at earlier. The most important is the classification of degrees coding models of $\mathrm{Th}(\mathbb{N})$.

Corollary 7.11: Let S be a Scott set containing the arithmetic sets. There is $M \models \mathrm{Th}(\mathbb{N})$ with $\mathbb{R}\mathrm{e}(M) = S$ and $\deg(M) = d$ iff d enumerates S.

We can strengthen one of the Knight-Lachlan-Soare results using 7.11 and the following theorem of Hodes [Ho].

Theorem 7.12: (Hodes [Ho]). There is an enumeration E of the arithmetic sets such that $E'' \equiv_T 0^{(\omega)}$.

Corollary 7.13: There is $M \models \mathrm{Th}(\mathbb{N})$ s.t. $\mathbb{R}\mathrm{e}(M)$ is the set of arithmetic sets and $\deg(M)'' \equiv_T 0^{(\omega)}$.

7.10 may also be used to strengthen 1.27 and 2.1.

Corollary 7.14: The following are equivalent for any Scott set S.

i) There is a d-effective enumeration of S.

ii) There is a d enumeration of S.

iii) There is a complete extension T of PA s.t. $T \equiv d$ and $\mathrm{Rep}(T) = S$.

iv) For any $T \in S$ a complete extension, there is a recursively saturated $M \models T$ with $\mathrm{Re}(M) = S$ and $\deg(M) = \underset{\sim}{d}$.

Corollary 7.15: d enumerates a Scott set iff d separates a pair of effectively inseparable r.e. sets.

§8. DEGREES CODING MODELS OF PT

In this section we shift gears and examine degrees coding models of PT, a theory introduced by DeMillo and Lipton to study the polynomial time hierarchy.

<u>Definition 8.1:</u> Let L be the usual language of arithmetic augmented by function symbols for all polynomial time computable functions and relation symbols for all polynomial time decidable relations.

Let PT be the set of all $\exists\forall$ sentences true in the standard model.

In [D-L], DeMillo and Lipton relate classical questions about the polynomial time hierarchy to questions of provability in PT. For example: Suppose $S \in NP \cap coNP$, then there are polynomial time recognizable predicates $A(x,y)$ and $B(x,y)$ such that $x \in F$ iff $\exists y\, A(x,y)$ and $x \notin S$ iff $\exists y\, B(x,y)$. We say A and B represent S. Let $\Delta_S(A,B)$ be the sentence $\forall x(\exists y\, A(x,y) \hat{\vee} \exists z\, B(x,z))$.

<u>Theorem 8.2</u>: (DeMillo-Lipton [D-L]). Let $S \in NP \cap coNP$. Then $S \in P$ iff $PT \vdash \Delta_S(A,B)$ for some A,B representing S.

<u>Proof</u>: ($\Rightarrow$) If $S \in P$, there are $A(x)$ and $B(x)$ in L such that $x \in S$ iff $A(x)$ and $x \notin S$ if $B(x)$. Clearly $PT \vdash \forall x(\exists y\, A(x) \vee \exists z\, B(x))$.

($\Leftarrow$) Suppose $PT \vdash \Delta_S(A,B)$.

<u>Claim</u>: For some $f_1,\ldots f_m$ function symbols in L,

$$PT \vdash \forall x \bigvee_{i=1}^{m} (A(x,f_i(x)) \vee B(x,f_i(x))).$$

Suppose not. Then $\Gamma(v) = \{\neg A(v,f(v)) \wedge \neg B(v,f(v)) : f \in L\}$ is consistent with PT. Let $M \models PT$ with $c \in M$ realizing $\Gamma(v)$. Let M' be the substructure of M generated by c. PT is essentially a universal theory (all existential sentences are witnessed in the standard part and hence by terms in the language), so $M' \models PT$. But $M' \models \nexists y(A(c,y) \vee B(c,y))$, a contradiction.

Thus, $x \in S$ iff $\bigvee_{i=1}^{m} (A(x,f_i(x)) \vee B(x,f_i(x)))$, but this is polynomial time decidable. //

Similar arguments allow you to prove that for $S \in NP$, $PT \vdash$ "$S \in$ cO-NP" if $P = NP$.

DeMillo and Lipton hoped that the model theory for PT would be sufficiently tractable to produce nonstandard models. Solovay destroyed these hopes by showing that PT has no recursive nonstandard models. This forced DeMillo and Lipton to conclude that "PT is almost as strong as PA". McAloon conjectured that models of PT are much harder to produce than models of PA. We confirm this below by showing that if M is a nonstandard model of PT then the atomic diagram of the reduct of M to the language of arithmetic has Turing degree above $\underset{\sim}{0}'$. We begin by showing that $\mathbb{R}e(M)$ is a Scott set. We do this by producing nonstandard initial segments of M which are models of PA.

Fix M a nonstandard model of PT. Let $\log(x)$ be the polynomial time computable function $\log(x) = \max\{y : 2^y \leq x\}$. We think of log as a function from $M \to M$ by taking the interpretation of the function symbol in L for log. Let $c \in M$ be nonstandard. Let $N = \{m \in M: \text{ for every } n \in \omega \ \log^{(n)} c \geq m\}$, where $\log^{(n)} x$ denotes

$\underbrace{\log(\log(\log(\ldots\log}_{n \text{ times}}(x)).$

Lemma 8.3: (Solovay [D-L]). i) $N \supsetneq \omega$ and N is an initial segment of M.

ii) N is closed under the function $x \mapsto 2^x$.

iii) N is closed under all polynomial time computable functions.

Hence $N \models PT$.

Proof: i) Clearly N is an initial segment of M and $N \supseteq \omega$. So we need only show N contains a nonstandard element. If $x \in M$, let $|x|$ denote the length of a binary representation of x. Define a function

$$\ell(n,x) = \begin{cases} x & \text{if } n = 0 \\ |\ell(n-1,x)| & \text{if } n > 0. \end{cases}$$

Define $\ell^*(n,x) = \ell(|n|,x)$, $\ell^*(n,x)$ is polynomial time computable. As $c \notin \omega$, for every $n \in \omega$, $\ell^*(n,c) > 1$. Define $R(x,y) \iff \ell^*(x,y) = 1$. It is easily seen that $\forall y\, R(|y|,y)$. Thus $\forall y\, \exists x \le y\, R(x,y)$. Define $f_R(y)$ to be the least $v < |y|$ so that $R(v,y)$, f_R is polynomial time computable. Let θ be the sentence $\forall x \forall y (R(f_R(y),y) \wedge (x < f_R(y) \rightarrow -R(x,y)))$. As $N \models \theta$, $M \models \theta$. Let $f_R(c) = d$. Note $d > \omega$. We see that c is of the order

$$\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\} \text{-d times.}$$

So $\log^{(n)} c$ is of the order

$$\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\} \text{-d-n times.}$$

As d is nonstandard $[d/2] < d-n$ for $n \in \omega$, where $[x]$ is the greatest integer less than x. Thus for each n, $[d/2] < d-n < \log^{(n)} c$. So $[d/2] \in N \setminus \omega$.

ii) If $a < \log^{(n+1)} c$, $2^a < \log^{(n)} c$. So if $a \in N$, then $2^a \in N$. (Note: 1) We are implicitly using the fact that $\{(x,y): x = 2^y\}$ is polynomial time decidable. 2) In general, models of PT are not closed under exponentiation (the model M' used in the proof of 8.2 gives an example of one which is not).)

iii) Let f be computable in time n^k. Then for any x, $|f(x)| \le |x|^k$. So $f(x) \le 2^{|x|^k} \le x^{|x|^k}$. As $x^{|x|^k}$ is dominated by 2^x, for any nonstandard a, $f(a) \le 2^a$. Thus by ii), if $a \in N$, $f(a) \in N$. //

<u>Lemma 8.4</u>: $N \models I\Sigma_0^*$, where $I\Sigma_0^*$ denotes the theory with an induction schema for bounded formulas in the language L.

<u>Proof</u>: We begin with an illustrative example which avoids the notation obscurities of the general case. Let $P(x,y,\bar{z})$ be a polynomial time decidable predicate. Let $\bar{a} \in N$ and suppose $N \models \exists x\, \forall y < x\; P(x,y,\bar{a})$. Let $b \in N$ such that $\forall y < b\; P(b,y,\bar{a})$. We claim there is a least d with this property.

We define a new predicate $R(w,\bar{z})$ by $\forall y < |\omega|\; P(|w|,y,\bar{z})$. $R(w,\bar{z})$ is polynomial time decidable and $N \models R(2^b,\bar{a})$. Define $Q(v,\bar{z})$ by $\exists s < |v|\; R(s,\bar{z})$. Q is polynomial time decidable and $N \models Q(2^{2^b},\bar{a})$.

Define $f(u,\bar{z}) = \begin{cases} 0 & \neg Q(u,\bar{z}) \\ \text{least } s\; R(s,\bar{z}) & Q(u,\bar{z}). \end{cases}$

f is polynomial time computable.

Let θ be the sentence $\forall u,\bar{z},x(\neg Q(u,\bar{z}) \vee (Q(u,\bar{z}) \wedge R(f(u,\bar{z}),\bar{z}) \wedge (x < f(u,\bar{z}) \rightarrow \neg R(x,\bar{z}))$. Then $\mathbb{N} \models \theta$, so $N \models \theta$. Let $d = f(2^{2^b},\bar{a})$. Then d is least such that $\forall y < |d|\ P(|d|,y,\bar{a})$. So $|d|$ is the least x such that $\forall y < x\ P(x,y,z)$.

We now extend these ideas to the general case. Suppose $N \models \exists x\ \varphi(x,\bar{a})$, where $\varphi(x,\bar{z})$ is Σ_0 in the language L. Let $b \in N$ such that $\varphi(b,\bar{a})$. We will show there is a least d such that $\varphi(d,\bar{a})$.

We write $\varphi(x,\bar{z})$ as $Q_0v_0 \leq t_0 \ldots Q_nv_n \leq t_n$, $P(x,\bar{v},\bar{z}_0,\bar{z}_1)$, where each t_i is either x or an element of $\bar{z}_0$, $\bar{z}_0$ and $\bar{z}_1$ are disjoint with union $\bar{z}$ and $P(x,\bar{v},\bar{z}_0,\bar{z}_1)$ is polynomial time decidable. Break $\bar{a}$ up into $\bar{a}_0$ and $\bar{a}_1$ in the obvious way.

Define $R(y,\vec{w},\vec{z}_1)$ by $Q_0v_0 \leq |s_0|.\ Q_nv_n \leq |s_n|$, $P(|y|,\bar{v},|\bar{w}|_1\bar{z}_1)$ where, if $t_i = x$, then $s_i = y$ and if $t_i = (\bar{z}_0)_j$, then $s_i = w_j$. R is polynomial time decidable and $N \models R(2^b,2^{\bar{a}_0},a_i)$, where $2^{\bar{a}_0}$ denotes $<2^{i_0},2^{i_1}\ldots 2^{i_h}>$ where $\bar{a}_0$ is $<i_0,\ldots i_K>$. Define $Q(v,\bar{w},\bar{z}_1)$ by $\exists s < |v|\ R(s,\vec{w},\bar{z}_1)$. Q is polynomial time computable and $N \models Q(2^{2^b},2^{a_0},\bar{a}_1)$. Define a Skolem function

$$f(u,\bar{w},\bar{z}_1) = \begin{cases} 0 & \neg Q(u,\bar{w},\bar{z}_1) \\ \text{the least } s \text{ s.t. } R(s,\bar{w},\bar{z}_1) & Q(u,\bar{w},\bar{z}). \end{cases}$$

f is polynomial time computable. Let θ be the sentence

$$\forall u,\bar{z}_1,\bar{w},x\ [\neg Q(u,\bar{w},\bar{z}_1) \vee (Q(u,\bar{w},\bar{z}_1) \wedge R(f(u,\bar{w},\bar{z}_1),\bar{w},\bar{z}_1) \wedge (x < f(u,\bar{w},\bar{z}_1) \rightarrow \neg R(x,\bar{w},\bar{z}_1)))].$$

$\mathbb{N} \models \theta$, so $N \models \theta$. Let $d = f(2^{2^b},a^{a_0},\bar{a}_1)$. Then d is minimal such that $R(d,2^{a_0},\bar{a}_1)$. So $|d|$ is minimal such that $\varphi(|d|,\bar{a})$. Thus $N \models I\Sigma_0^*$.

//

In particular, $\bar{N}$ the reduct of N to the language of arithmetic is a model of $I\Sigma_0$. This will allow us to use the following theorem of McAloon.

Theorem 8.5: (McAloon [Mc]). If $M \models I\Sigma_0$ is nonstandard, there is a nonstandard initial segment of M modeling PA.

Fix an initial segment I of M such that $I \models PA$. We show $\mathbb{R}e(M)$ is a Scott set by showing $\mathbb{R}e(M) = \mathbb{R}e(I)$.

Lemma 8.6: If $a \in M$ and $b \in M$ is nonstandard, there is $d < b$ such that $r(a) = r(d)$.

Proof: Let $F(n) = \max\{x : x! \leq n\}$. Let $\varphi(x,y)$ be the formula $x < y \to \exists z < |x|\ \forall w < F(|x|) \forall v < w\ \ (v \text{ prime} \to (v|y \leftrightarrow v|z))$. It is easily verified that $\varphi(x,y)$ is polynomial time computable. As $\mathbb{N} \models \forall x,y\ \varphi(x,y)$, so does M. In particular, $M \models \varphi(b,a)$. Thus, there is $d < |b|$ such that $\forall w < F(|b|) \forall v < w$ $(v \text{ prime} \to (v|a \leftrightarrow v|d))$. As $F(|b|)$ is nonstandard, for $n \in \omega$ $p_n|d \leftrightarrow p_n|a$. So $r(a) = r(d)$. //

Corollary 8.7: $\mathbb{R}e(M) = \mathbb{R}e(I)$, so $\mathbb{R}e(M)$ is a Scott set.

Corollary 8.8: (Solovay). If $M \models PT$ is nonstandard, $\mathrm{Diag}(M) \;_T\!> 0$.

Here we use the fact that $r(a) \leq_T \mathrm{Diag}(M)$ because the Euclidean algorithm works for sufficiently small numbers.

Lemma 8.9: $0' \in \mathbb{R}e(M)$.

Proof: Let $f(\bar{x}) = 0$ be a Diophantine equation. $M \models \exists \bar{x} f(\bar{x}) = 0$ iff $\mathbb{N} \models \exists \bar{x} f(\bar{x}) = 0$. Further, if $\mathbb{N} \models \exists \bar{x} f(\bar{x}) = 0$, then $I \models \exists \bar{x} f(\bar{x}) = 0$ and if $I \models \exists \bar{x} f(\bar{x}) = 0$, then $M \models \exists \bar{x} f(\bar{x}) = 0$, so $\mathbb{N} \models \exists \bar{x} f(\bar{x}) = 0$. Hence

I and $\mathbb{N}$ have the same Diophantine theory. By Martijasevic's theorem [D-M-R], $Th_{\Sigma_1}(I) = Th_{\Sigma_1}(\mathbb{N})$. By 1.17, $Th_{\Sigma_1}(I) \in \mathbb{R}e(I) = \mathbb{R}e(M)$. But $Th_{\Sigma_1}(\mathbb{N}) \equiv_T 0'$. //

Putting everything together we get the following theorem which proves McAloon's conjecture.

Theorem 8.10: If $M \models PT$ is nonstandard and $\bar{M}$ is the reduct of M to the language of arithmetic, then there is a Diag(M) enumeration of a Scott set containing $0'$.

We can prove a converse to 8.10.

Lemma 8.11: Let S be a Scott set containing $0'$. There is a recursively saturated $M \models PT$ with $\mathbb{R}e(M) = S$.

Proof: Let $T_0 = \{\exists\bar{x}\ P(\bar{x}) : \mathbb{N} \models \exists\bar{x}\ P(\bar{x})\} \cup \{\forall\bar{x} : P(\bar{x}) : \mathbb{N} \models P(\bar{x})\}$. Clearly, $T_0 \leq_T 0'$. Suppose $M \models T_0^*$ and $PT \vdash \exists\bar{x}\ \forall\bar{y}\ P(\bar{x},\bar{y})$. Then there is $\bar{n} \in \omega$ such that $\mathbb{N} \models \forall\bar{y}\ P(\bar{n},\bar{y})$. Then $\forall\bar{y}\ P(\bar{n},\bar{y}) \in T_0^*$. So $M \models \exists\bar{x}\ \forall\bar{y}\ P(\bar{x},\bar{y})$. Hence $M \models PT$.

Let $C = \{c_0, c_1, c_2, \ldots\}$ be a set of Henkin constants. Let $\tau_1(\bar{v}), \tau_2(\bar{v})\ldots$ list all partial types with finitely many constants from C which are coded in S. Let $\varphi_0, \varphi_1, \varphi_2 \ldots$ list all $L(C)$ sentences. We build $T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$ so that each $T_i \in S$ and each T_i has finitely many constants.

Step 0: $T_0 = T_0$.

Step $s = 4_{n+1}$: Let $T_s = T_{s-1} \cup \{\pm\varphi_n\}$ to maintain consistency.

<u>Step $s = 4_n+2$</u>: If we have added $\exists x\, \psi(x)$ at step $s-1$ pick a new constant c and let $T_s = T_{s-1} \cup \{\psi(c)\}$.

<u>Step $s = 4_n+3$</u>: If $\tau_n(\bar{v})$ is consistent with T_{s-1}, pick a new c and let $T_s = T_{s-1} \cup \tau_n(c)$.

<u>Step $s = 4_n+4$</u>: Let T_s be a completion of T_{s-1} inside S, adding no new constants.

Let M be the Henkin model of $\cup T_n$. Step 4_n+3 insures M is recursively saturated and $\mathbb{R}\mathrm{e}(M) \supseteq S$. Step 4_n+4 insures $\mathbb{R}\mathrm{e}(M) \subseteq S$. //

<u>Theorem 8.12</u>: If d enumerates a Scott set S containing $0'$, then there is a recursively saturated $M \models PT$ with $\mathbb{R}\mathrm{e}(M) = S$ and $\mathrm{Diag}(M) \equiv_T d$.

<u>Proof</u>: Clear from 8.11 and 5.13. //

APPENDIX

Proposition A1: There is $M \models I\Sigma_0$ and a nonstandard $a \in M$ such that $[0,a]$ is not recursively saturated.

We define a theory P_{top} in the language of arithmetic, with the following axioms.

- $\forall x\ s(x) \neq 0$
- $\exists ! x\ s(x) = x$ (we denote this as α)
- $\forall x\ \forall y((x \neq \alpha \wedge y \neq \alpha \wedge s(x) = s(y)) \rightarrow x = y)$
- $\forall x(x+0 = x)$ $\qquad \forall x\ \forall y(x+s(y) = s(x+y))$
- $\forall x(x \cdot 0 = 0)$ $\qquad \forall x\ \forall y(x \cdot s(y) = x \cdot y + x)$
- $\forall \bar{x}((\varphi(0,\bar{x}) \wedge \forall y(\varphi(y,\bar{x}) \rightarrow \varphi(y+1,\bar{x})) \rightarrow \forall y\ \varphi(y,\bar{x}))$.

P_{top} is the natural thory of structures $[0,a]$ where $a \in M$ and M is a model of a reasonably rich arithmetic theory, like $I\Sigma_0$. Paris showed that in fact, every model of P_{top} arises this way.

Lemma A2: (Paris [G-M-W]). Every model $\mathfrak{A}$ of P_{top} admits an end extension $\mathfrak{B}$ to a model of $I\Sigma_0$.

$\mathfrak{B}$ is obtained in a natural way by considering polynomials over $\mathfrak{A}$.

To prove A1 then, it is sufficient to show that P_{top} has a nonrecursively saturated model. But this is easy. The induction axioms of P_{top} guarantee that every model has definable Skolem functions. Thus, if $\mathfrak{A} \models P_{top}$, we can take $\mathfrak{B} \prec \mathfrak{A}$ pointwise definable. $\mathfrak{B}$ cannot be recursively saturated as we omit the type $\Gamma(v) = \{\varphi(v) \rightarrow \exists w \neq v \varphi(w) : \varphi\}$. Thus, by Paris's lemma, there is $M \models I\Sigma_0$ with $a \in M$ such that $\mathfrak{B} \cong [0,a]$.

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