Math 435 Number Theory I

Problem Set 1

Due: Friday September 2

All work should be shown. Proofs should be written clearly in complete sentences.

1) a) Use the Euclidean algorithm to calculate $\gcd(1785, 1309)$.

b) Find integers x and y such that $1785x + 1309y = \gcd(1785, 1309)$.

2) Decide if the following equations have integer solutions. If so, find all integer solutions.

a) $165X + 45Y = 5$

b) $165X + 45Y = 30$

c) $165X - 45Y = 15$

3) a) Suppose $a, b, c \neq 0$. Prove that $\gcd(a, b, c) = \gcd(\gcd(a, b), c)$.

b) Let a, b, c be nonzero integers. Prove that the equation

$$aX + bY + cZ = d$$

has a integer solutions if and only if $\gcd(a, b, c)|d$.

c) Find a solution to the equation

$$14X + 35Y + 10Z = 243.$$

4) (Efficency of Euclidean algorithm)

a) Prove that if $a > b > 0$ and $a = bq + r$ where $0 \leq r < b$, then $r < \frac{a}{2}$

b) Suppose $r_{-1} > r_0 > r_1 > r_2 > \ldots > r_{n-1} = r_n > 0$ is the sequence of remainders obtained doing the Euclidean algorthim to find $\gcd(a, b)$ where $r_{-1} = a$ and $r_0 = b$. Prove that $r_i < \frac{r_{i-2}}{2}$ for $i = 1, \ldots, n$ and that $r_{2i} < \frac{b}{2^i}$ for $1 \leq i$ and $2i \leq n$.

c) Conclude that the Euclidean algorithm terminates in at most $2\log_2(b)$ steps.

d) Roughtly, how many steps will be need to find $\gcd(a, b)$ when b has 100 digits?